國立高雄大學 103 學年度第 1 學期應用數學系博士班資格考試試題

科目：組合數學　　日期: 103 年 9 月 24 日　　考試時間：180 分鐘

You need to show all your work.

1. (10 points) Prove the recurrence $\begin{bmatrix} n \\ k \end{bmatrix}_q = q^{n-k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}_q + \begin{bmatrix} n-1 \\ k \end{bmatrix}_q$.

2. (10 points) Find the generating function $F(z)$ with $[z^n]F(z) = \sum_k \binom{r}{k}\binom{r}{n-2k}$.

3. (10 points) What combinatorial significance has the n-th coefficient in the product $\prod_{k\geq 1}(1+z^k)$?

4. (10 points) Let h_n be the number of ordered set partitions of $\{1,\ldots,n\}$. Compute $\sum_{n\geq 0} h_n \frac{z_n}{n!}$.

5. (15 points) A permutation $\sigma \in S(n)$ is an *involution* if $\sigma^2 = \mathrm{id}$, that is, if all cycles have length 1 or 2. Prove for the number i_n of involutions the recurrence $i_{n+1} = i_n + n i_{n-1}$ $(i_0 = 1)$. What is the number of fixed-point-free involutions?

6. (15 points) In an election, exactly n persons vote for candidate A and n people for candidate B. They throw their ballots into the ballot box one after the other. Show that the number of possible ballot lists in which at any stage the number of votes for A is at least as large as that for B equals the Catalan number C_n.

7. (15 points) Interpret $\zeta^k(a,b)$ for a poset P, and compute ζ^k for the chain $\mathbb{C}_n$ and the Boolean poset $\mathbb{B}(n)$.

8. (15 points) How many patterns do we obtain when we color the 12 edges of a cube with 2 different colors?