國立高雄大學 108 學年度第 2 學期應用數學系博士班資格考試試題

科目：高等線性代數　　日期：109 年 3 月 11 日　　考試時間：180 分鐘

Notation.

$M_{m\times n}(\mathbb{R})$: the set of $m\times n$ real matrices.

$A\in M_{n\times n}(\mathbb{R})$ is positive definite if $A=A^\top$ and $x^\top Ax>0$ for all $x\neq 0$.

1 Let $A\in M_{n\times n}(\mathbb{R})$ and $W=\{B\in M_{n\times n}(\mathbb{R})|AB=BA\}$.

a. (5%) Show that W is a vector space.

b. (7%) Let $A=\mathrm{diag}\{2,1,3,2\}$ be a diagonal matrix. Find a basis for W.

c. (10%) Let $A=\begin{bmatrix}3&1&1\\2&4&2\\-1&-1&1\end{bmatrix}$. Find a basis for W.

2 (28%) Let $v_1=(1,2,3)^\top$, $v_2=(1,1,1)^\top$, $v_3=(1,0,1)^\top$, $W_1=\mathrm{span}\{v_1\}$ and $W_2=\mathrm{span}\{v_2,v_3\}$. Find $A\in M_{3\times 3}(\mathbb{R})$ such that $T=L_A$.

a. $T:\mathbb{R}^3\to\mathbb{R}^3$ is the projection on W_1 along W_2.

b. $T:\mathbb{R}^3\to\mathbb{R}^3$ is the projection on W_2 along W_1.

c. $T:\mathbb{R}^3\to\mathbb{R}^3$ is the reflection about W_2.

d. $T:\mathbb{R}^3\to\mathbb{R}^3$ is linear and $T(v_1)=e_1$, $T(v_2)=e_2$ and $T(v_3)=e_3$, where $\{e_1,e_2,e_3\}$ is the standard basis for $\mathbb{R}^3$.

3 (10%) Determine whether $\lim_{m\to\infty}A^m$ exists for each of the following matrices A, and compute the limit if it exists

a. $A=\begin{bmatrix}0.1&0.7\\0.7&0.1\end{bmatrix}$　　b. $A=\begin{bmatrix}0.4&0.7\\0.6&0.3\end{bmatrix}$

4 Let $A\in M_{n\times n}(\mathbb{R})$ be a symmetric matrix.

a. (10%) Show that every eigenvalues of A is real.

b. (10%) Show that there is an orthogonal matrix Q such that $Q^\top AQ$ is diagonal.

c. (10%) If $A\in M_{n\times n}(\mathbb{R})$ is positive definite if and only if there exists an $n\times n$ invertible matrix B such that $A=B^\top B$.

5 (10%) Show that if $A\in M_{m\times n}(\mathbb{R})$ is of rank n, there exists $B\in M_{n\times m}(\mathbb{R})$ such that $BA=I_n$.