國立高雄大學 109 學年度第 1 學期應用數學系博士班資格考試試題

科目：高等線性代數　　日期：109 年 10 月 7 日　　考試時間：180 分鐘

Notations:

$\mathbb{M}_n(\mathbb{C})$: the set of $n \times n$ complex matrices.

A^* : the conjugate transpose of an $m \times n$ matrix A.

1. (15%) Let V be the set of all real numbers in the form
$$a + b\sqrt{2} + c\sqrt{5},$$
where a, b, and c are rational numbers. Show that V is a vector space over the rational number filed $\mathbb{Q}$. Find $\dim(V)$ and a basis of V.

2. (15%) Let $A, B \in \mathbb{M}_n(\mathbb{C})$. if $B[A\ I] = [I\ B]$, show that $B = A^{-1}$. Explain why A^{-1}, if it exists, can be obtained by row operations.

3. (15%) Prove that the eigenvectors belonging to different eigenvalues are linearly independent.

4. (10%) Let $A \in \mathbb{M}_n(\mathbb{C})$. Show that for any unit vector $x \in \mathbb{C}^n$,
$$|x^* A x|^2 \leq x^* A^* A x.$$

5. (10%) Find the eigenvalues of
$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 4 & 3 & 4 \\ 1 & 2 & 1 & 2 \\ 3 & 4 & 3 & 4 \end{bmatrix}$$

6. (10%) Let A and B be $m \times n$ matrices such that $B^* A = 0$. Show that
$$\operatorname{rank}(A + B) = \operatorname{rank}(A) + \operatorname{rank}(B) \leq n.$$

7. (10%) Let $A \in \mathbb{M}_n(\mathbb{R})$ be a symmetric matrix. Given an unit vector $u \in \mathbb{R}^n$. Find $\theta \in \mathbb{R}$ such that $\|Au - \theta u\|_2 = \min_{t \in \mathbb{R}} \|Au - tu\|_2$.

8. (15%) A square matrix A is said to be nilpotent if for some positive integer k
$$A^k = 0.$$
Prove that A is nilpotent if and only if all the eigenvalues of A are zero.