國立高雄大學一〇一學年度第二學期應用數學研究所博士班資格考試試題

科目：實變函數論　　日期: 102 年 2 月 26 日　　考試時間：180 分鐘

(1) (10 %) Let (X, d) be a metric space. Show that, for any open subset G of X, it can be expressed as a countable union of closed subsets of X.

(2) (10 %) If $\{r_k\}$ denotes the rational numbers in $[0, 1]$ and $\{a_k\}$ satisfies $\sum |a_k| < +\infty$, show that $\sum a_k |x - r_k|^{-1/2}$ converges absolutely a.e. in $[0, 1]$.

(3) (15 %) Let $\mathcal{C}$ be the space of continuous functions on $[-1, 1]$ with the supremum norm. Show that there is no $g \in L^1[-1, 1]$ such that $f(0) = \int_{-1}^{1} f(t)g(t)\, dt$ for every $f \in \mathcal{C}$.

(4) (10 %) Let $(X, \mathcal{F}, \mu)$ be a finite measure space and $f \in L^1(X, \mu)$ be complex-valued. Assume that there is a closed set S in the complex plane such that the averages $\frac{1}{\mu(E)} \int_E f d\mu$ lie in S for every $E \in \mathcal{F}$ with $\mu(E) > 0$. Show that $f(x) \in S$ for almost all $x \in X$.

(5) (10 %) Let $1 \le p \le \infty$, $f \in L^p(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$. Show that $f * g \in L^p(\mathbb{R}^n)$ and $\|f * g\|_p \le \|f\|_p \cdot \|g\|_1$, where $f * g(x) = \int_{\mathbb{R}^n} f(x - t)g(t)\, dt$, $x \in \mathbb{R}^n$.

(6) (a) (10 %) Let (Ω, Σ, μ) be a measure space and $f \in L^1(\Omega, \Sigma, \mu)$. Show that the set function $F(A) = \int_A f\, d\mu$, $A \in \Sigma$, is absolutely continuous.

(b) (10 %) Let f be an absolutely continuous function on $\mathbb{R}$ such that $f(0) = 0$ and its derivative is square-integrable with respect to the Lebesgue measure. Show that

$$\lim_{t \to \pm\infty} \frac{f(t)}{\sqrt{1 + t^2}} = 0.$$

(7) (10 %) Let μ be a probability measure on a measurable space Ω. Suppose that f and g are positive measurable functions on Ω such that $f \cdot g \ge 1$. Prove that

$$\int_\Omega f\, d\mu \cdot \int_\Omega g\, d\mu \ge 1.$$

(8) (15 %) Let f be an integrable function over the set E with respect to the measure μ. If $E_n = \{x \in E : |f(x)| \ge n\}$, $n = 1, 2, \ldots$, prove that $n \cdot \mu(E_n) \to 0$ as $n \to \infty$.