國立高雄大學九十九學年度第一學期應用數學研究所博士班資格考試試題

科目：實變函數論　　　　日期: 99 年 9 月 30 日　　　　考試時間：180 分鐘

(1) (a) (10 %) Let $\{f_n\}$ be a sequence of measurable functions of the measurable space (Ω, Σ, μ) such that $\int_\Omega \sum_n |f_n|\, d\mu < +\infty$. Show that $\int_\Omega (\sum_n f_n)\, d\mu = \sum_n \int_\Omega f_n\, d\mu$.

(b) (12 %) Show that $\int_0^\infty \frac{\sin ax}{e^x - 1}\, dx = \sum_{n=1}^\infty \frac{a}{n^2+a^2}$. (Hint: using (a) and the inequality $|\sin u| \le |u|$ for any $u \in \mathbb{R}$).

(2) (10 %) Use Fubini's theorem to prove that $\int_{\mathbb{R}^n} e^{-|\vec{x}|^2}\, d\vec{x} = \pi^{\frac{n}{2}}$.

(3) (12 %) Let f be a real-valued function on $\mathbb{R}$. Suppose that $f''(0)$ exists and is finite. Show that

$$f''(0) = \lim_{h \to 0} \frac{f(h) - 2f(0) + f(-h)}{h^2}.$$

(4) (12 %) Let $1 \le p \le \infty$, $f \in L^p(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$. Show that $f * g \in L^p(\mathbb{R}^n)$ and $\|f * g\|_p \le \|f\|_p \cdot \|g\|_1$, where $f * g(x) = \int_{\mathbb{R}^n} f(x-t) g(t)\, dt$, $x \in \mathbb{R}^n$.

(5) (a) (10 %) Let (Ω, Σ, μ) be a measure space and $f \in L^1(\Omega, \Sigma, \mu)$. Show that for any $\varepsilon > 0$, there is $\delta > 0$ such that if $A \in \Sigma$ with $\mu(A) < \delta$, then $|\int_A f\, d\mu| < \varepsilon$.

(b) (12 %) Let $1 \le p < +\infty$ and $f \in L^p(\mathbb{R})$. For all $x \in \mathbb{R}$, set $F(x) = \int_0^x f(t)\, dt$. Show that

$$\exists\, M > 0 \ni \forall\, x \in \mathbb{R},\ \limsup_{h \to 0} \frac{|F(x+h) - F(x)|}{|h|^{1-\frac{1}{p}}} \le M.$$

(6) (12 %) Let (Ω, Σ, μ) be a finite measure space. Suppose $f \in L^\infty(\Omega, \mu)$, $\|f\|_\infty > 0$, and $\alpha_n = \int_\Omega |f|^n\, d\mu$ $(n = 1, 2, 3, \ldots)$. Prove that $\lim\limits_{n \to \infty} \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_\infty$.

(7) (15 %) Let K be a function defined on $\mathbb{R}^2$ that is Lebesgue measurable and such that

$$A^2 = \iint_{\mathbb{R}^2} |K(x,y)|^2\, dxdy < +\infty.$$

For every function $f \in L^2(\mathbb{R})$, the function Tf is defined by

$$Tf(x) = \int_{\mathbb{R}} K(x,y) f(y)\, dy, \quad \forall\, x \in \mathbb{R}.$$

Show that (a) Tf is Lebesgue measurable and (b) $\|Tf\|_2 \le A \cdot \|f\|_2$.