國立高雄大學九十九學年度第二學期應用數學系博士班資格考試試題

科目：實變函數論　　日期：100 年 2 月 21 日　　考試時間：180 分鐘

1. (10%) Let $(X, \mathcal{F}, \mu)$ be a measure space $\{E_n\} \subset \mathcal{F}$.

   (a) If $\{E_n\}$ is a sequence of increasing measurable set, show that $\lim_{n\to\infty} E_n$ exists and

   (*) $$\mu(\lim_{n\to\infty} E_n) = \lim_{n\to\infty} \mu(E_n).$$

   (b) Is the identity(*) true if $\{E_n\}$ is a decreasing sequence of sets?

2. (10%) Let $\{f_n\}$ and f be measurable functions defined on $[0,1]$. Show that f_n converges to f in measure iff every subsequence of $\{f_n\}$ has in turn a subsequence that converges almost everywhere to f.

3. (10%) If $p > 0$ and $\int_E |f - f_k|^p \to 0$ as $p \to \infty$ show that there is a subsequence $f_{k_j} \to f$ a.e. in E.

4. (10%) Let $(X, \mathcal{B}, \mu)$ be a measure space and $f \in L^1(\mu)$.

   (a) For any $E \in \mathcal{B}$, define a set function ν by $\nu(E) = \int_E f\, d\mu$, show that ν is a signed measure.

   (b) Show that to each $\epsilon > 0$ there exist a $\delta > 0$ such that $\int_E |f| d\mu < \epsilon$ whenever $\mu(E) < \delta$.

5. (10%) Let μ be a Borel measure on $\mathbb{R}^1$ and ν a probability measure on $\mathbb{R}^1$. Suppose that $\int_{\mathbb{R}^1} g(y)\, d\nu(y) = 0$. Let φ be a convex function on $\mathbb{R}^1$.Prove that, for any convex function φ on $\mathbb{R}^1$,

$$\iint_{\mathbb{R}^2} \varphi(f(x) + g(y))\, d\mu(x) d\nu(y) \geq \int_{\mathbb{R}^1} \varphi(f(x))\, d\mu(x),$$

   provided that $\varphi(f) \in L^1(\mu)$.

6. (15%) Assume that $\|f\|_1 < \infty$. Show that $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$.

7. (15%) Suppose that f, g are C^1-functions such that $g(\pm\infty) = 0$.

   (a) If $\int_{-\infty}^{\infty} f'(x) g(x)\, dx$ converges as an improper integral in the sense of Riemann integral, show that $\int_{-\infty}^{\infty} f(x) g'(x)\, dx$ also converges and the following integration by parts formula holds

$$\int_{-\infty}^{\infty} f'(x) g(x)\, dx = \int_{-\infty}^{\infty} f(x) g'(x)\, dx$$

   (b) If $f' g \in L^1(\mathbb{R}^1)$ and $\int_{-\infty}^{\infty} f g'$ exists in the sense of Lebesgue integral. Show that the above integration by parts formula also holds.

8. (10%) Let $(X, \mathcal{F}, \mu)$ be a measure space and f be a non-negative measurable function. Define

$$\omega(t) = \mu\{x : f(x) > t\}.$$

   Show that, for any $p > 0$,

$$\int_X f^p(x) d\mu(x) = -\int_0^\infty t^p\, d\omega(t).$$

9. (10%) If f is a bounded integrable C^1-function on $(\mathbb{R}^1)$ and $g \in L^1(\mathbb{R}^1)$, show that $f * g$ is a bounded integrable C^1 function on $\mathbb{R}^1$.