國立高雄大學九十八學年度第一學期應用數學系博士班資格考試試題

科目：實變函數論　　　　日期: 99 年 1 月 20 日　　　　考試時間：180 分鐘

(1) Evaluate the following limits:

(a) (7 %) $\lim_{n\to\infty} \int_{-\infty}^{+\infty} (1 - e^{-\frac{t^2}{n}}) e^{-|t|} \sin^3 t\, dt$

(b) (8 %) $\lim_{\epsilon\to 0} \int_0^{\infty} (1 - e^{(\epsilon x)^2}) e^{-x^3} \sin^4 x\, dx$

(2) (10 %) Use Fubini's theorem and the relation $\frac{1}{x} = \int_0^{\infty} e^{-xt}\, dt$ $(x > 0)$ to prove that

$$\lim_{A\to\infty} \int_0^A \frac{\sin x}{x}\, dx = \frac{\pi}{2}.$$

(3) (10 %) Let $(X, \mathcal{F}, \mu)$ be a finite positive measure space. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of $\mathbb{R}$-valued measurable functions. Show that $f_n \to 0$ in measure as $n \to \infty$ iff $\int_X \frac{|f_n(x)|}{1+|f_n(x)|}\, d\mu(x) \to 0$ as $n \to \infty$.

(4) (10 %) Let f be defined on $[0, 1]$ by $f(x) = x^2 \sin(x^{-2})$ for $0 < x \leq 1$ and $f(0) = 0$. Prove that f has finite derivative at all points of $(0, 1)$ and $f'_+(0) = 0$. Is f continuous, uniformly continuous, absolutely continuous on $[0, 1]$? Is f of bounded variation on $[0, 1]$?

(5) (10 %) Let $(X, \mathcal{F}, \mu)$ be a finite positive measure space. If $\{f_n\}$ is a sequence of a.e. real-valued measurable functions in X, show that there exists positive constants λ_n such that $\{f_n/\lambda_n\}$ converges a.e. to zero.

(6) (10 %) Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Suppose $f \in L^{\infty}(X, \mu)$, $\|f\|_{\infty} > 0$, and $\alpha_n = \int_X |f|^n\, d\mu$ $(n = 1, 2, 3, \ldots)$. Prove that $\lim_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_{\infty}$.

(7) (10 %) Let $(X, \mathcal{F}, \mu)$ be a positive measure space, $f, \{f_k\} \in L^p(X, \mu)$. If $f_k \to f$ a.e. and $\|f_k\|_p \to \|f\|_p$, $1 \leq p < \infty$, show that $\|f - f_k\|_p \to 0$. Is this true for $L^{\infty}(X, \mu)$?

(8) (10 %) Show that if $f \in L^1(\mathbb{R}, \lambda)$ and $f = f * f$, then $f = 0$ a.e., where λ is the Lebesgue measure and $*$ is the convolution of functions. (Hint: Use the Fourier transform)

(9) Let $(X, \mathcal{F}, \mu)$ be a positive measure space. Call a set $\Phi \subset L^1(X, \mu)$ uniformly integrable if for each $\epsilon > 0$, corresponds a $\delta > 0$ such that $|\int_E f\, d\mu| < \epsilon$ whenever $f \in \Phi$ and $\mu(E) < \delta$.

(a) (7 %) Prove that every finite subset of $L^1(X, \mu)$ is uniformly integrable.

(b) (8 %) If $\mu(X) < \infty$, $\{f_n\}$ is uniformly integrable, $f_n \to f$ a.e. as $n \to \infty$, and $|f(x)| < \infty$ a.e., show that $\lim_{n\to\infty} \int_X |f_n - f|\, d\mu = 0$. (Hint: Use Egoroff's theorem)