Abstract Algebra

Time allowed: 3 hours.

(12 points each)

(1) Let $\mathfrak{S}_7$ be the group of permutations of seven objects. Find all n such that some element of $\mathfrak{S}_7$ has order n.

(2) Factor $x^4 + x^3 + x + 3$ completely in $\mathbb{Z}_5[x]$.

(3) Let I denote the ideal in $\mathbb{Z}[x]$ generated by $x^3 + x + 1$ and 5. Is I a prime ideal? Why?

(4) Let G be a finite group with identity e. Suppose that for every $a, b \in G$ distinct from e, there is an automorphism σ of G such that $\sigma(a) = b$. Prove that G is abelian.

(5) Let G be a group and H a subgroup of index $n < \infty$. Prove or disprove the following:

(a) If $a \in G$, then $a^n \in H$.

(b) If $a \in G$, then for some k, $1 \leq k \leq n$, we have $a^k \in H$.

(6) (a) Find the group of automorphisms $\mathrm{Aut}(\mathbb{Z})$ of $\mathbb{Z}$.

(b) Find the group of automorphisms $\mathrm{Aut}(\mathfrak{S}_3)$ of the symmetric group $\mathfrak{S}_3$.

(7) Let R be a ring with identity, and let I be the left ideal of R generated by

$$\{ab - ba : a, b \in R\}.$$

Prove that I is a two-sided ideal.

(8) Let F be a finite field of cardinality p^n, with p prime and $n > 0$, and let G be the group of invertible 2×2 matrices with coefficients in F. Prove that the G has order

$$(p^{2n} - 1)(p^{2n} - p^n).$$

(9) Classify all groups (up to isomorphism) of order 12.

(10) Prove that if p is a prime number, then the polynomial

$$f(x) = x^{p-1} + x^{p-2} + \cdots + 1$$

is irreducible in $\mathbb{Q}[x]$.