國立高雄大學 106 學年度第 1 學期應用數學系博士班資格考試試題

科目：高等線性代數　　日期：106 年 9 月 29 日　　考試時間：180 分鐘

Notation.

$M_{m\times n}(\mathbb{R})$: the set of $m\times n$ real matrices.

1 Let $A\in M_{n\times n}(\mathbb{R})$. Define $W_A=\{B\in M_{n\times n}(\mathbb{R})|AB=BA\}$.

a. (5) Show that W_A is a vector space.

b. (7) Let $A=\begin{bmatrix}1 & 1\\ 4 & 1\end{bmatrix}$. Find a basis for W_A.

2 (10) Let $A\in M_{m\times n}(\mathbb{R})$ and $B\in M_{n\times m}(\mathbb{R})$. Suppose that $\lambda\neq 0$ is an eigenvalue of AB then show that λ is an eigenvalue of BA.

3 Let $\mathcal{O}=\{Q\in M_{n\times n}(\mathbb{R})|\ Q \text{ is orthogonal}\}$.

a. (8) Show that $\mathcal{O}$ is a group under matrix multiplication.

b. (5) Suppose that $Q\in\mathcal{O}$ and $x\in\mathbb{R}^n$. Show that $\|Qx\|_2=\|x\|_2$.

c. (5) Let $x\in\mathbb{R}^n$ with $\|x\|_2=1$. Show that $H_x=I_n-2xx^T$ is orthogonal.

d. (10) Let $b=[1,1,0,1]^\top\in\mathbb{R}^4$. Find a unit vector $x\in\mathbb{R}^4$ such that $H_xb=\sqrt{3}[1,0,0,0]^\top$, where $H_x=I_n-2xx^T$.

4 a. (10) Show that $A\in M_{n\times n}(\mathbb{R})$ is a positive definite matrix if and only if there exists an invertible matrix $B\in M_{n\times n}(\mathbb{R})$ such that $A=BB^\top$.

b. (5) Suppose that A is positive definite and $Ax=y$. Show that $x^\top y>0$.

c. (10) Suppose that $x,y\in\mathbb{R}^n$ and $x^\top y>0$. Show that there is a positive definite matrix A such that $Ax=y$.

5 Let $A=\begin{bmatrix}8 & 10\\ -5 & -7\end{bmatrix}$.

a (8) Find the general solution of the system of differential equations

$$x^{'}(t)=Ax(t).$$

b. (7) Let $F(x)=\frac{Ax}{\|Ax\|_2}$ and $x_0=[1,1]^\top$. Find $\lim_{n\to\infty}F^n(x_0)$.

6 (10) Let $A,\ B\in M_{n\times n}(\mathbb{R})$. If B is invertible, then show that there exist a scalar $\lambda\in\mathbb{C}$ and a nonzero vector $x\in\mathbb{C}^n$ such that $Ax=\lambda Bx$.