Linear Algebra

1. (3×10 points) Let $\mathbf{A} \in M_{n\times n}(\mathbb{R})$. Suppose $\mathbf{A}$ is diagonalizable and $f(t)$ is the characteristic polynomial of $\mathbf{A}$.

   (a) Show that $f(\mathbf{A}) = \mathbf{0}$.

   (b) If $p(t)$ is the minimal polynomial of $\mathbf{A}$, show that $p(t)$ divides $f(t)$.

   (c) What can you conclude on $\dim(\mathrm{span}(\mathbf{I}, \mathbf{A}, \mathbf{A}^2, \ldots))$?

2. (2×10 points) Let $\mathbf{A}$ be a 5×5 real matrix and $f(t) = (t-1)^2(t-2)^2$. Suppose $f(\mathbf{A}) = \mathbf{0}$.

   (a) Find all possible Jordan forms of $\mathbf{A}$.

   (b) If f is the minimal polynomial of $\mathbf{A}$, find all possible Jordan forms of $\mathbf{A}$.

3. (3×10 points) Let $\mathbf{u} = (1,1,1)^\top \in \mathbb{R}^3$.

   (a) Find an othonormal basis for $\mathbb{R}^3$ that concludes $\mathbf{u}/\|\mathbf{u}\|$.

   (b) Recall $\{\mathbf{u}\}^\perp$ denotes the hyperplane in $\mathbb{R}^3$ that is perpendicular to $\mathbf{u}$. Find the matrix representation of the reflection about $\{\mathbf{u}\}^\perp$.

   (c) Find the rotation matrix about the line $\{t\mathbf{u} |\ t \in \mathbb{R}\}$, orientated with the direction $\mathbf{u}$, counter-clockwise through an angle θ.

4. (2×10 points) Let $\mathbf{A} \in M_{n\times n}(\mathbb{R})$ be a symmetric matrix with distinct eigenvalues
$$\lambda_1 < \lambda_2 < \cdots < \lambda_n;$$
and let $\mathbf{u} \in \mathbb{R}^n$. Consider the $(n+1) \times (n+1)$ matrix
$$\mathbf{B} = \left[\begin{array}{c|c} 1 & \mathbf{u}^\top \\ \hline \mathbf{u} & \mathbf{A} \end{array}\right].$$

   (a) Show that roots of the equation
$$(1-\mu) + \mathbf{u}^\top(\mu\mathbf{I} - \mathbf{A})^{-1}\mathbf{u} = 0$$
   are eigenvalues of $\mathbf{B}$.

   (b) If $\mathbf{u}$ is nonzero and is NOT an eigenvector of $\mathbf{A}$, denoting $\mu_1 \le \mu_2 \le \cdots \le \mu_{n+1}$ eigenvalues of $\mathbf{B}$, show that
$$\mu_1 < \lambda_1 < \mu_2 < \lambda_3 < \cdots < \lambda_{n-1} < \mu_n < \lambda_n < \mu_{n+1}.$$