國立高雄大學一百學年度第一學期應用數學系博士班資格考試試題

科目：實變函數論　　日期: 100 年 9 月 26 日　　考試時間：180 分鐘

1. (36%) Prove or disprove the following propositions:

   1.1 Let μ be a signed measure and f a measurable function. If, for any measurable sets E, we have $\int_E f(x)\mu(dx) = 0$, then $f = 0$ a.e. (μ).

   1.2 If f is integrable with respect to a measure μ, then the set $S := \{x : f(x) \neq 0\}$ is σ-finite.

   1.3 Let $\{f_n\}$ be a sequence of measurable functions which converges to 0 a.e. with respect to the measure μ. Then f_n must converges to 0 in measure.

   1.4 Convergence in L^p, $(p > 0)$ implies convergence almost everywhere.

   1.5 $l^p \subset l^\infty$ for any $p > 0$.

   1.6 l^∞ is separable.

2. (10%) Let $(X, \mathcal{B}, \mu)$ be a measure space. Show that $L^p \cap L^q \subset L^r$ if $r \in (p, q)$. In particular, show that $L^r \subset L^1 \cap L^\infty$.

3. (10%) Let f_n be a sequence of function such that $f_n \geq g$ a.e., where $g \in L^p$. Suppose that $\|f_n\|_p \leq M$ and $f_n \to f$ a.e.. Show that $f \in L^p$.

4. (10%) Let $(X, \mathcal{B}, \mu)$ be a measure space and $f \in L^1(\mu)$. Show that,

$$\lim_{\mu(E)\to 0} \int_E f\, d\mu = 0.$$

5. (10%) Let μ be a Borel measure on $\mathbb{R}^1$ and ν a probability measure on $\mathbb{R}^1$. Suppose that $\int_{\mathbb{R}^1} g(y)\, d\nu(y) = 0$. Show that, for any $p \geq 1$,

$$\iint_{\mathbb{R}^2} |f(x) + g(y))|^p\, d\mu(x) d\nu(y) \geq \int_{\mathbb{R}^1} |f(x)|^p\, d\mu(x),$$

   provided that $f \in L^p(\mu)$.

6. (12%) Evaluate

$$\lim_{p\to\infty} \left\{ \int_0^{\pi/2} \left| \frac{x}{\sin x} \right|^p dx \right\}^{1/p}.$$

7. (12%) If f, g are integrable functions on $(\mathbb{R}^1)$ and show that $f * g$ is integrable and

$$\|f * g\|_1 \leq \|f\|_1 \|g\|_1.$$