國立高雄大學 101 學年度第 1 學期應用數學系博士班資格考試試題

科目：實變函數論　　日期：101 年 9 月 28 日　　考試時間：180 分鐘

PHD Qualification Examination (Real Analysis)

September 28, 2012

1. (10%) Given any set $A \subset \mathbb{R}^1$, show that there is a $\mathcal{G}_\delta$−set G such that $m^*(A) = m^*(G)$.

2. (10%) Let $\{f_n\}$ be a sequence of nonnegative functions on $\mathbb{R}^1$ such that $f_n \to f$ a.e., and suppose that $\int f_n \to \int f < \infty$. Then for any measurable set E,
$$\int_E f_n \to \int_E f.$$

3. (20%) Let T is a Lipschitz function on $\mathbb{R}^1$ and f a continuous function.
   (1) Show that f map F_σ−type sets into F_σ−type sets.
   (2) Show that T sends sets of measure zero into sets of measure zero.
   (3) Show that T maps measurable sets into measurable sets.

4. (20%) Show that $\|f\|_p \to \|f\|_\infty$ as $p \to \infty$, provided that there exists some r such that $\|f\|_r < \infty$. (Hint: Show that $\limsup_{p\to\infty} \|f\|_p \le \|f\|_\infty \le \liminf_{p\to\infty} \|f\|_p$.)

5. (10%) If f is a bounded continuous function and $g \in L^1$, show that $f * g$ is also a bounded continuous function.

6. (10%) Let f be a function such that, for all $g \in L^p$ ($p \ge 1$), $\int |f||g| < \infty$. Show that $f \in L^q$, where $1/p + 1/q = 1$

7. (20%) Suppose $1 < p < \infty$, $f \ge 0$ and $f \in C_c(0,\infty)$ and
$$F(x) = \frac{1}{x}\int_0^\infty f(t)\,dt \qquad (0 < x < \infty)$$
   Prove that
$$\|F\|_p \le \frac{p}{p-1}\|f\|_p.$$
   (Hint: Apply integration by parts formula to $\int_0^\infty F^p(x)\,dx$.)

8. (10%) Show that the inequality in Problem 7 remains true if $f \in L^p(0,\infty)$.