國立高雄大學一〇二學年度第二學期應用數學研究所博士班資格考試試題

科目：實變函數論 日期: 103 年 2 月 26 日 考試時間：180 分鐘

(1) (a) (10 %) Let (Ω, Σ, μ) be a measure space, and let $\{f_n\}$, $\{g_n\}$ be two sequences of measurable functions on Ω. Assume that (i) $|f_n| \leq g_n$ a.e. for all n (ii) $f_n \to f$ and $g_n \to g$ a.e. (iii) $g_n \in L^1(\Omega, \mu)$ with $\int_\Omega g_n(x)\,\mu(dx) \to \int_\Omega g(x)\,\mu(dx)$. Show that $f \in L^1(\Omega, \mu)$ and $\lim_{n\to\infty} \int_\Omega f_n(x)\,\mu(dx) = \int f(x)\,\mu(dx)$.

✓(b) (10 %) If $\{r_k\}$ denotes the rational numbers in $[0,1]$ and $\{a_k\}$, a sequence of real numbers, satisfies $\sum_k |a_k| < +\infty$, show that $\sum_k a_k|x - r_k|^{-1/2}$ converges absolutely a.e. in $[0,1]$.

(2) ✓(a) (10 %) Use Fubini's theorem and the relation $\frac{1}{x} = \int_0^\infty e^{-xt}\,dt$ $(x > 0)$ to prove that $\lim_{A\to\infty} \int_0^A \frac{\sin x}{x}\,dx = \frac{\pi}{2}$.

(b) (10 %) Show that $\int_0^\infty \frac{1-\cos x}{x^2}\,dx = \frac{\pi}{2}$.

(3) (10 %) Show that for $f \in L^1([0, 2\pi], \lambda)$,

$$\lim_{k\to\infty} \int_0^{2\pi} f(x) \cos kx\,dx = 0,$$

where λ is the Lebesgue measure.

(4) (10 %) Show that if $f \in L^\infty([0,1], \lambda)$ and for all n in $\mathbb{N} \cup \{0\}$, $\int_0^1 t^n f(t)\,dt = 0$ then $f = 0$ a.e., where λ is the Lebesgue measure.

(5) Let $(X, \mathcal{F}, \mu)$ be a measure space.

(a) (10 %) For $\{E_k\} \subset \mathcal{F}$, if $\sum_k \mu(E_k) < +\infty$, show that $\mu(\limsup E_k) = 0$.

✓(b) (10 %) Assume that $\mu(X) < +\infty$. If $\{f_n\}$ is a sequence of a.e. real-valued measurable functions in X, show that there exists positive constants λ_n such that $\{f_n/\lambda_n\}$ converges a.e. to zero.

✓(6) (10 %) Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Suppose $f \in L^\infty(X, \mu)$, $\|f\|_\infty > 0$, and $\alpha_n = \int_X |f|^n\,d\mu$ $(n = 1, 2, 3, \ldots)$. Prove that $\lim\limits_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_\infty$.

✓(7) (10 %) Let f_k, f be in $L^p(\mathbb{R}^n)$, $1 < p < \infty$. Suppose that $f_k \to f$ a.e. and that $\sup_k \|f_k\|_p \leq M < +\infty$. Show that $\int f_k g \to \int fg$ for all $g \in L^q(\mathbb{R}^n)$, $1/p + 1/q = 1$.