國立高雄大學 104 學年度第一學期應用數學研究所博士班資格考試試題

科目：實變函數論　　日期: 104 年 9 月 30 日　　考試時間：180 分鐘

(1) (10 %) If $\{r_k\}$ denotes the rational numbers in $[0,1]$, show that $\sum_k k^{-2}|x-r_k|^{-1/2}$ converges absolutely a.e. in $[0,1]$.

(2) (10 %) Let $(X,\mathcal{F},\mu)$ be a finite measure space, and $\mathcal{G}$ be a sub-$\sigma$-algebra of $\mathcal{F}$. Suppose $f \in L^2(X,\mathcal{F},\mu)$. Show that there exists a $g \in L^2(X,\mathcal{G},\mu)$ such that for any $h \in L^2(X,\mathcal{G},\mu)$, $\int_X (f(x)-g(x))h(x)\,\mu(dx) = 0$.

(3) (15 %) Let $(X,\mathcal{F},\mu)$ be a measure space. Suppose $f \in L^1(X,\mathcal{F},\mu)$. Prove that to each $\epsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f|\,d\mu < \epsilon$ whenever $\mu(E) < \delta$.

(4) (10 %) Show that if $f \in L^\infty([0,1],\lambda)$ and for all $n$ in $\mathbb{N}\cup\{0\}$, $\int_0^1 t^n f(t)\,dt = 0$ then $f = 0$ a.e., where $\lambda$ is the Lebesgue measure.

(5) (15 %) Let $(X,\mathcal{F},\mu)$ be a finite measure space, and $\{f_n\}$ a sequence of $\mathcal{F}$-measurable functions in $X$. Show that $f_n \to 0$ in measure if and only if $\int_X \frac{|f_n|}{1+|f_n|}\,d\mu \to 0$.

(6) (10 %) Let $(X,\mathcal{F},\mu)$ be a measure space. Show that for $\{E_k\} \subset \mathcal{F}$, if $\sum_k \mu(E_k) < +\infty$, show that $\mu(\limsup E_k) = 0$.

(7) (15 %) (10 %) Let $(X,\mathcal{F},\mu)$ be a probability space. Suppose that $f$ and $g$ are positive measurable functions on $X$ such that $f \cdot g \geq 1$. Prove that

$$\int_X f\,d\mu \cdot \int_X g\,d\mu \geq 1.$$

(8) (15 %) Let $(X,\mathcal{F},\mu)$ be a finite measure space. Suppose $f \in L^\infty(X,\mu)$, $\|f\|_\infty > 0$, and $\alpha_n = \int_X |f|^n\,d\mu$ $(n = 1,2,3,\ldots)$. Prove that $\lim\limits_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_\infty$.