國立高雄大學九十八學年度第一學期應用數學系博士班資格考試試題

科目：高等線性代數　　　　　日期: 99 年 1 月 20 日　　　　　考試時間：180 分鐘

There are 10 problems, 10 points each. Please write down your answers clearly and carefully. Time allowed: 3 hours.

1. (a) Find the characteristic polynomial, minimal polynomial, rational canonical form, Jordan canonical form J of the matrix $\begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

   (b) Find a matrix M such that $J = M^{-1}AM$.

2. (a) Let W be a a finite-dimentional vector space and W_1 and W_2 its subspaces. Prove that $\dim W_1 + \dim W_2 = \dim(W_1 + W_2) + \dim(W_1 \cap W_2)$.

   (b) Prove that $\dim U + \dim W = \dim V$ if $0 \to U \to V \to W \to 0$ is an exact sequence of finite dimensional vector spaces.

3. Let $\mathbb{P}_3$ be the vector space of all real polynomials of degree 3 or less. Define the inner product $\langle f, g\rangle := \int_0^1 fg\,dx$. Find an orthonormal basis of $\mathbb{P}_3$.

4. (a) Let A be an $n \times n$ real matrix and $v_1, \ldots v_n \in \mathbb{R}^n$ be the column vectors of A. Show that $|\det(A)| \leq ||v_1|| \cdot ||v_2|| \ldots ||v_n||$.

   (b) Let M be a positive symmetric $n \times n$ real matrix with diagonal entries $d_1, \ldots, d_n$. Show that $\det(M) \leq d_1 \cdot d_2 \ldots d_n$.

5. Let A, B be $n \times n$ matrices over a field $\mathbb{F}$. Suppose that A is invertible.

   (a) If $\mathbb{F}$ has characteristic 0, prove that there exists $\lambda \in \mathbb{F}$ such that $\lambda A + B$ is invertible.

   (b) Give an example to show that the conclusion of part (a) can fail when $\mathbb{F} = \mathbb{Z}/2\mathbb{Z}$.

6. For any $n \times n$ complex matrix A let $e^A := \sum_{k=0}^{\infty} \frac{1}{k!} A^k$.

   (a) Prove that $e^{B^{-1}AB} = B^{-1}e^A B$ for any invertible complex $n \times n$ matrix B.

   (b) Prove that $\det(e^A) = e^{\text{trace}(A)}$

7. Let V be an n-dimensional subspace of the space of polynomials over $\mathbb{C}$. Show that there exists $f_1, \ldots, f_n \in V$ and an integer ℓ such that $f_i(\ell + j) = \delta_{ij}$, $i, j = 1, 2, \ldots, n$.

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8. Show that the number of invertible $n \times n$ matrices over $\mathbb{F}_p$ is

$$\prod_{i=0}^{n=1} (p^n - p^i),$$

where p is a prime number and $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$.

9. How many 3×3 real orthogonal matrices are there, up to similarity, with the property $\det(A) = -1$ and $A^3 = A$?

10. Let A be an $n \times n$ complex matrix and V the vector space of $n \times n$ complex symmetric matrices. Let $L : V \to V$ be the linear transformation defined by $L(X) = AXA^t$. Suppose that A is diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_n$. Find the eigenvalues of L.