國立高雄大學九十九學年度第二學期應用數學系博士班資格考試試題

科目：高等線性代數　　　日期: 100 年 2 月 21 日　　考試時間：180 分鐘

1. (20 points) For
$$\mathbf{A} = \begin{bmatrix} 1 & 4 \\ 2 & 3 \end{bmatrix} \in M_{2\times 2}(\mathbb{R}),$$
find an expression for $\mathbf{A}^n$, where n is an arbitrary integer.

2. (10 points) Suppose $\beta = \{\mathbf{u}_1, \ldots, \mathbf{u}_k\}$ is an orthonormal set (with respect to the standard inner product) in $\mathbb{R}^n$. Let $W = \text{span } \beta$ and $\mathbf{T}$ be the orthonormal projection on W. Find $\mathbf{A} \in M_{n\times n}(\mathbb{R})$ such that $\mathbf{T}(\mathbf{x}) = \mathbf{A}\mathbf{x}$. Specify your answer.

3. (10 points) Let $f(x, y) = x^2 + 4xy + 4y^2$. Find new coordinates x', y' so that $f(x, y)$ can be written as $\lambda_1(x')^2 + \lambda_2(y')^2$. Determine the graph of $f(x, y) = 1$ is an ellipse or a hyperbola.

4. (20 points) A linear operator $\mathbf{T}$ on a finite dimensional real inner product space is called *positive definite* if $\mathbf{T}$ is self-adjoint and $\langle \mathbf{T}(\mathbf{x}), \mathbf{x}\rangle > 0$ for all $\mathbf{x}$. A matrix $\mathbf{A} \in M_{n\times n}(\mathbb{R})$ is *positive definite* if $L_{\mathbf{A}}$ is positive definite. Now, let $\mathbf{T}$ be self-adjoint and $\mathbf{A} = [\mathbf{T}]_\beta$, where β is an orthonormal basis of V. Show that $\mathbf{T}$ is positive-definite if and only if $\sum_{i,j} a_i a_{ij} a_j > 0$ for all n-tuples $(a_1, \ldots, a_n)$.

5. (20 points) Let
$$\mathbf{A} = \begin{bmatrix} 0 & -3 & 1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & 1 & -1 & 2 \\ -2 & -3 & 1 & 4 \end{bmatrix}.$$
The characteristic polynomial of $\mathbf{A}$ is given by $t^4 - 4t^3 + 4t^2$. Find a Jordan form $\mathbf{J}$ for $\mathbf{A}$ and a matrix $\mathbf{Q}$ such that $\mathbf{J} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$.

6. (20 points) For $\mathbf{A} \in M_{n\times n}(\mathbb{R})$, let $q_{\mathbf{A}}(t)$ denote the minimal polynomial of $\mathbf{A}$ and $\mathcal{P}(\mathbf{A})$ denote the vector space
$$\{p(\mathbf{A}) \mid p \text{ is any real coefficient polynomial}\}.$$
Show that $\dim \mathcal{P}(\mathbf{A}) = \deg q_{\mathbf{A}}(t)$.