國立高雄大學一百學年度第二學期應用數學系博士班資格考試試題

科目：組合數學　　日期：101 年 2 月 29 日　　考試時間：180 分鐘

1. (15%) Solve the following recurrence relation.
   (a) (distinct real roots) $a_n = 5a_{n-1} - 6a_{n-2}$, with initial conditions $a_0 = 2$, $a_1 = 1$.
   (b) (root with multiplicity) $a_n = 6a_{n-1} - 9a_{n-2}$, with initial conditions $a_0 = a_1 = 1$.
   (c) (distinct complex roots) $a_n = 2a_{n-1} - 2a_{n-2}$, with initial conditions $a_0 = 3, a_1 = -2$.
   (d) (inhomogeneous) $a_n = 3a_{n-1} + 2n$, with initial conditions $a_0 = 1$.
2. (20%)
   (a) State the definition of the *Stirling number of the first kind* $s_{n,k}$. Calculate $s_{5,3}$. Prove that
   $$s_{n,k} = s_{n-1,k-1} + (n-1)s_{n-1,k}.$$
   (b) State the definition of the *Stirling number of the second kind* $S_{n,k}$. Calculate $S_{5,3}$. Prove that
   $$S_{n,k} = S_{n-1,k-1} + kS_{n-1,k}.$$
3. (15%) We say $\pi \in \mathfrak{S}_n$ is a *derangement* if $\pi(i) \neq i$ for $1 \leq i \leq n$.
   (a) Find the number D_n of derangements in $\mathfrak{S}_n$.
   (b) Prove the identity $D_n = (n-1)(D_{n-1} + D_{n-2})$.
   (c) Prove the recurrence $D_n = nD_{n-1} + (-1)^n$.
4. (20%)
   (a) State the "*Lagrange inversion formula*".
   (b) Use the 'Lagrange inversion formula' to derive the number of binary trees on n nodes.
5. (10%) Let $p(n;k)$ be the number of partitions of n with exactly k parts, prove that
   $$p(n;k) = p(n-1;k-1) + p(n-k;k).$$
6. (10%) Find the generating function
   $$F(z) = \sum_{n\geq 0} \binom{2n}{n} z^n$$
   of the central binomial numbers $\binom{2n}{n}$.
7. (10%) Prove that
   $$\sum_{k=0}^{n} \binom{s+k}{k}\binom{n-k}{m} = \binom{s+n+1}{n-m}.$$