國立高雄大學一〇二學年度第一學期應用數學研究所博士班資格考試試題

科目：實變函數論　　　　　日期: 102 年 9 月 25 日　　　　　考試時間：180 分鐘

(1) (15 %) Evaluate
$$\lim_{n\to\infty} \int_{[0,n]} \left(1-\frac{x}{n}\right)^n e^{\frac{x}{2}}\, dx.$$

(2) (20 %) For a positive function $f$ on $\mathbb{R}^n$, let $D_f = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} :\ 0 \le t \le f(x)\}$ and $D'_f = \{(x,t) \in \mathbb{R}^n \times \mathbb{R} :\ 0 \le t < f(x)\}$.

(a) Show that if $f$ is measurable, then, for $p > 0$,
$$\int_{\mathbb{R}^n} f(x)^p\, dx = p\int_0^\infty t^{p-1}\mu_n(\{f > t\})\, dt = p\int_0^\infty t^{p-1}\mu_n(\{f \ge t\})\, dt,$$
where $\mu_k$ denotes the Lebesgue measure on $\mathbb{R}^k$ for any $k \in \mathbb{N}$.

(b) Show that if $f$ is measurable, $\mu_{n+1}(D_f) = \mu_{n+1}(D'_f)$ and the graph of $f$ is a set of measure zero in $\mathbb{R}^{n+1}$.

(3) (15 %) Let $f_k, f$ be in $L^p(\mathbb{R}^n)$, $1 < p < \infty$. Suppose that $f_k \to f$ a.e. and that $\sup_k \|f_k\|_p \le M < +\infty$. Show that $\int f_k g \to \int fg$ for all $g \in L^q(\mathbb{R}^n)$, $1/p + 1/q = 1$.

(4) (15 %) Does there exist a continuous real valued function $f(x)$, $0 \le x \le 1$, such that
$$\int_0^1 xf(x)\, dx = 1 \quad \text{and} \quad \int_0^1 x^n f(x)\, dx = 0$$
for $n = 0, 2, 3, \ldots$? Give an example or a proof that no such $f$ exists.

(5) (10 %) Let $1 \le p \le \infty$, $f \in L^p(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$. Show that $f * g \in L^p(\mathbb{R}^n)$ and $\|f * g\|_p \le \|f\|_p \cdot \|g\|_1$, where $f * g(x) = \int_{\mathbb{R}^n} f(x-t)g(t)\, dt$, $x \in \mathbb{R}^n$.

(6) (15 %) Let $(\Omega, \Sigma, \mu)$ be a finite measure space. Let $\{f_n\}$ be a sequence of $\mathbb{R}$-valued measurable functions. Show that $f_n \to 0$ in measure as $n \to \infty$ if and only if
$$\int_\Omega \frac{|f_n(x)|}{1+|f_n(x)|}\, d\mu(x) \to 0$$
as $n \to \infty$.

(7) (10 %) Let $(\Omega, \Sigma, \mu)$ be a finite measure space. Suppose $f \in L^\infty(\Omega, \mu)$, $\|f\|_\infty > 0$, and $\alpha_n = \int_\Omega |f|^n\, d\mu$ $(n = 1, 2, 3, \ldots)$. Prove that $\lim\limits_{n\to\infty} \frac{\alpha_{n+1}}{\alpha_n} = \|f\|_\infty$.