國立高雄大學一〇四學年度第二學期應用數學研究所博士班資格考試試題

科目：實變函數論　　　　日期: 105 年 3 月 9 日　　　　考試時間：180 分鐘

(1) (10 %) Let X be a topological space and let μ be a positive Borel measure on X such that every point has a neighborhood which has a finite μ-measure. Show that every compact set is a finite μ-measure set.

(2) (10 %) Let (X, d) be a metric space. Show that, for any open subset G of X, it can be expressed as a countable union of closed subsets of X.

(3) (10 %) Let H be an infinite-dimensional Hilbert space and μ be a finite Borel measure on H. Show that μ is not translation invariant.

(4) (10 %) Let $(X, \mathcal{F}, \mu)$ be a finite measure space and $f \in L^1(X, \mu)$ be complex-valued. Assume that there is a closed set S in the complex plane such that the averages $\frac{1}{\mu(E)} \int_E f d\mu$ lie in S for every $E \in \mathcal{F}$ with $\mu(E) > 0$. Show that $f(x) \in S$ for almost all $x \in X$.

(5) (10 %) Let $1 \leq p \leq \infty$, $f \in L^p(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$. Show that $f * g \in L^p(\mathbb{R}^n)$ and $\|f * g\|_p \leq \|f\|_p \cdot \|g\|_1$, where $f * g(x) = \int_{\mathbb{R}^n} f(x-t)g(t)\, dt$, $x \in \mathbb{R}^n$.

(6) (10 %) Let (Ω, Σ, μ) be a measure space and $f \in L^1(\Omega, \Sigma, \mu)$. Show that the set function $F(A) = \int_A f\, d\mu$, $A \in \Sigma$, tends to zero as the measure of A tends to zero.

(7) (15 %) Let $(X, \mathcal{F}, \mu)$ be a finite measure space. Suppose that $f_k \to f$ μ-a.e. and that $f_k, f \in L^p(X, \mu)$, $1 < p < \infty$. If $\|f_k\|_p \leq M < +\infty$, show that $\int_X f_k g\, d\mu \to \int_X f g\, d\mu$ for all $g \in L^q(X, \mu)$, $1/p + 1/q = 1$.

(8) (10 %) Let μ and ν be two probability measures on the Borel σ-algebra of $\mathbb{R}$ and f, g be two real-valued Borel measurable functions on $\mathbb{R}$. Suppose that $\int_{\mathbb{R}} g(y)\, d\nu(y) = 0$. Prove that, for $p \geq 1$,

$$\iint_{\mathbb{R}^2} |f(x) + g(y)|^p\, d\mu(x) d\nu(y) \geq \int_{\mathbb{R}} |f(x)|^p\, d\mu(x).$$

(9) (15 %) Let f be an integrable function over the set E with respect to the measure μ. If $E_n = \{x \in E : |f(x)| \geq n\}$, $n = 1, 2, \ldots$, prove that $n \cdot \mu(E_n) \to 0$ as $n \to \infty$.