國立高雄大學 103 學年度第 1 學期應用數學系博士班資格考試試題

科目：高等線性代數　　　日期：103 年 9 月 24 日　　　考試時間：180 分鐘

Linear Algebra

1. (10+10 points) Let $\mathbf{T}$ be a linear operator on an n-dimensional vector space V over a field F, let β be an ordered basis for V, and let $\mathbf{A} = [\mathbf{T}]_\beta$. We define the linear transformation $\phi_\beta : V \to F^n$ by $\phi_\beta(\mathrm{v}) = [\mathrm{v}]_\beta$.

   (a) If $\mathbf{v} \in V$ and $\phi_\beta(\mathrm{v})$ is an eigenvector of $\mathbf{A}$ corresponding to the eigenvalue λ, then $\mathbf{v}$ is an eigenvector of $\mathbf{T}$ corresponding to the eigenvalue λ.

   (b) If λ is an eigenvalue of $\mathbf{A}$ (and hence of T), then a vector $\mathbf{y} \in F^n$ is an eigenvalue of $\mathbf{A}$ corresponding to λ if and only if $\phi_\beta^{-1}(\mathbf{y})$ is an eigenvector of $\mathbf{T}$ corresponding to λ.

2. (10 points) Let $\mathbf{A}$ be an $n \times n$ matrix. Prove that

$$\dim(\mathrm{span}(\mathbf{I}, \mathbf{A}, \mathbf{A}^2, \ldots)) \le n.$$

3. (20 points) A linear operator $\mathbf{T}$ on a finite dimensional real inner product space is called *positive definite* if $\mathbf{T}$ is self-adjoint and $\langle \mathbf{T}(\mathrm{x}), \mathrm{x}\rangle > 0$ for all x. Prove $\langle \mathrm{x}, \mathrm{y}\rangle' = \langle \mathbf{T}(\mathrm{x}), \mathrm{y}\rangle$ defines another inner product on V.

4. (10 points) Prove that for any matrix $\mathbf{A} \in M_{m\times n}(\mathbb{C})$, $R(L_{\mathbf{A}^*})^\perp = N(L_{\mathbf{A}})$.

5. (20 points) Compute a QR-decomposition for the matrix $\mathbf{A} = \begin{pmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{pmatrix}$

6. (20 points) A linear operator $\mathbf{T}$ on a vector space V is called nilpotent if $\mathbf{T}^p = \mathrm{T}_0$ for some positive integer p. Here T_0 denotes the zero operator. An $n \times n$ matrix $\mathbf{A}$ is nilpotent if $\mathbf{A}^p = O$ for some positive integer p. Let $\mathbf{T}$ be a nilpotent operator on an n-dimensional vector space V. Suppose $\mathbf{T}^p = \mathrm{T}_0$. Prove that the characteristic polynomial of $\mathbf{T}$ is $(-t)^n$ in the following steps.

   (a) Prove that any upper triangular matrix with each diagonal entry equal to zero is nilpotent.

   (b) Show that $N(\mathbf{T}^i) \subseteq N(\mathbf{T}^{i+1})$ for every positive integer i.

   (c) There is a sequence of ordered basis $\beta_1, \ldots, \beta_p$ such that β_i is a basis of $N(\mathbf{T}^i)$ and $\beta_i \subseteq \beta_{i+1}$ for $1 \le i \le p-1$.

Linear Algebra

(d) Let $\beta = \beta_p$ be the ordered basis for $N(\mathbf{T}^p) = V$ in (c). Then $[\mathbf{T}]_\beta$ is an upper triangular matrix with each diagonal entry equal to zero.