國立高雄大學 109 學年度第 2 學期應用數學系博士班資格考試試題

科目：高等線性代數　　日期：110 年 3 月 12 日　　考試時間：180 分鐘

Notations:

$\mathbb{M}_n(\mathbb{R})$: the set of $n \times n$ real matrices.

1. (15%) Determine whether or not $\mathbb{R}^2$, with the operations

$$(x_1, y_1)^T + (x_2, y_2)^T = (x_1 x_2, y_1 y_2)^T$$

and

$$c(x_1, y_1)^T = (cx_1, cy_1)^T,$$

is a vector space over $\mathbb{R}$.

2. (15%) Show that if S is a real matrix and $S^T = -S$, then $I - S$ is nonsingular and the matrix $(I - S)^{-1}(I + S)$ is orthogonal.

3. (15%) Suppose $A \in \mathbb{R}^{n\times n}$ is an invertible square matrix and $u, v \in \mathbb{R}^n$ are column vectors. Then $A + uv^T$ is invertible if and only if $1 + v^T A^{-1} u \neq 0$. This is referred to as the Sherman-Morrison formula.

4. (10%) Show that a triangular orthogonal matrix is diagonal.

5. (10%) Let

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \text{ and } f(x) = \max_{x \neq 0} \frac{x^T A x}{x^T x}.$$

Find the maximum value of the function $f(x)$.

6. (10%) Show that if $A \in \mathbb{M}_n(\mathbb{R})$ and $A^2 = A$, then

$$\operatorname{rank}(A) + \operatorname{rank}(I - A) = n.$$

7. (10%) Let x and y be in $\mathbb{R}^n$ and define $\psi(\alpha) = \|x - \alpha y\|_2$. Find θ such that $\psi(\theta) = \min_{\alpha\in\mathbb{R}} \psi(\alpha)$.

8. (15%) Let A and B be an $m \times n$ and $n \times m$ matrices, respectively. Show that

$$\det(AB) = 0 \text{ if } m > n.$$