國立高雄大學 107 學年度第 1 學期應用數學系博士班資格考試試題

科目：高等線性代數　　日期：107 年 9 月 28 日　　考試時間：180 分鐘

Notation.

$M_{m\times n}(\mathbb{R})$: the set of $m \times n$ real matrices.

1 Let $A \in M_{n\times n}(\mathbb{R})$ be invertible and $u,\ v \in \mathbb{R}^n$.

a. (8%) Find the eigenvalues and eigenvectors of $uv^\top$.

b. (10%) Show that $v^\top A^{-1}u \neq -1$ if and only if $A + uv^\top$ is invertible.

c. (7%) Let $\delta = 1 + v^\top A^{-1}u \neq 0$. Then show that

$$(A + uv^\top)^{-1} = A^{-1} - \delta^{-1}A^{-1}uv^\top A^{-1}.$$

2 Let $A,\ B,\ X \in M_{n\times n}(\mathbb{R})$ and $AX - XB = 0$

a. (5%) Show that $p(A)X - Xp(B) = 0$ for every polynomial $p(t)$.

b. (10%) If A and B have no common eigenvalues, then $X = 0$.

c. (5%) If A and B have no common eigenvalues, then for each $C \in M_{n\times n}(\mathbb{R})$, show that $AX - XB = C$ has a unique solution.

3 **a.** (5%) State Schur's theorem.

b. (5%) Apply Schur's theorem to prove that if A is normal, then there a unitary matrix U such that $U^H AU$ is diagonal.

4 Let $\mathbf{e} = [1,1,1]^\top \in \mathbb{R}^3$. Define $T : \mathbb{R}^3 \to \mathbb{R}^3$ by $T(x) = \mathbf{e} \times x$, where"$\times$" is the cross product.

a. (5%) Show that T is linear transformation.

b. (5%) Find the null space of T.

c. (5%) Let $W = \text{span}\{\mathbf{e}\}^\perp$. Show that W is a T-invariant subspace.

d. (10%) Find the eigenvalues and eigenvectors of T^2.

5 Let $\angle(u,v)$ be the angle between vectors u and v.

a. (5%) Let $u,v \in \mathbb{R}^n$ with $\|u\|_2 = \|v\|_2 = 1$. If $u^\top v \geqslant 0$, then show that

$$\|u - v\|_2 = 2\sin(\frac{\angle(u,v)}{2}).$$

b. (5%) Let $\mathcal{V} = \text{span}\{v_1, v_2\}$ be a vector space, where $v_1 = [1,1,1,0]^\top$ and $v_2 = [1,1,1,1]^\top$. Suppose that $u_1 = [1,0,0,0]^\top$. Find

$$\min_{v\in\mathcal{V}} \angle(u_1, v).$$

c. (10%) Let $\mathcal{U} = \text{span}\{U\}$ and $\mathcal{V} = \text{span}\{V\}$, where $U, V \in M_{n\times\ell}(\mathbb{R})$ are orthogonal matrices. Show that

$$\max_{u\in\mathcal{U}} \min_{v\in\mathcal{V}} \angle(u,v) = \arccos(\sigma_{\max}),$$

where $\sigma_{\max}$ is the largest singular value of $V^\top U$.