國立高雄大學 108 學年度第 2 學期應用數學系博士班資格考試試題

科目：實變函數論　　　日期：109 年 3 月 11 日　　　考試時間：180 分鐘

In the following questions, we use $m^*(E)$ to denote the outer measure of E. If E is measurable, then we simply denote its measure by $m(E)$.

1. (10%) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function. Show that f' is Lebesgue measurable.

2. (10%) If E_1 and E_2 are subsets of $\mathbb{R}^n$, $E_1 \subset E_2$, E_1 is measurable, and $m(E_1) = m^*(E_2) < \infty$, show that E_2 is measurable.

3. (15%) If $p > 0$, $\int_E |f - f_k|^p \to 0$, and $\int_E |f_k|^p \le M$ for all k, show that

$$\int_E |f|^p \le M.$$

4. (15%) Let $1 \le p \le \infty$, $f \in L^p(\mathbb{R}^n)$ and $g \in L^1(\mathbb{R}^n)$. Prove that $f * g \in L^p(\mathbb{R}^n)$ and

$$\|f * g\|_p \le \|f\|_p \|g\|_1,$$

where $(f * g)(x) = \int_{\mathbb{R}^n} f(t) g(x - t)\, dt$.

5. (15%) Let g_k and g be integrable functions on E such that $g_k \to g$ a.e. in E. Let f_k and f be measurable functions such that $|f_k| \le g_k$ and $f_k \to f$ a.e. in E. If

$$\int_E g_k \to \int_E g,$$

show that

$$\int_E f_k \to \int_E f.$$

6. (15%) Let E be a measurable set in $\mathbb{R}^2$ and $E_x = \{y : (x, y) \in E\}$. If

$$m\Big(\Big\{x : m(E_x) = \frac{1}{2}\Big\}\Big) = 1,$$

show that

$$m\Big(\Big\{y : m(\{x : (x, y) \in E\}) \ge \frac{1}{2}\Big\}\Big) > 0.$$

7. (20%) Suppose that $f_k \to f$ a.e. and that f_k, $f \in L^p(\mathbb{R}^n)$, $1 < p < \infty$. If $\|f_k\|_p \le M < \infty$, show that

$$\int_{\mathbb{R}^n} f_k g \to \int_{\mathbb{R}^n} f g$$

for all $g \in L^{p'}(\mathbb{R}^n)$, $1/p + 1/p' = 1$.