國立高雄大學 109 學年度第 1 學期應用數學系博士班資格考試試題

科目：實變函數論　　　日期：109 年 10 月 7 日　　　考試時間：180 分鐘

1. (10%) **Prove or disprove** that the pointwise supremum of a family of lower semicontinuous functions is lower semicontinuous.

2. (10%) Suppose $f \in L^1(\mathbb{R}^n)$. **Prove or disprove** that given $\varepsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f| < \varepsilon$ whenever $|E| < \delta$, where $|E|$ denotes the Lebesgue measure of E.

3. (10%) For $n \in \mathbb{N}$, let $f_n : [0,1] \to \mathbb{R}$ be defined by $f_n(x) = \frac{nx^{n-1}}{1+x}$ for all $x \in [0,1]$. Show that

$$\lim_{n\to\infty} \int_0^1 f_n(x)\,dx = \frac{1}{2}.$$

4. (10%) Show that

$$\int_0^\infty \left(\int_0^r e^{-xy^2} \sin x\,dx \right) dy = \int_0^r \left(\int_0^\infty e^{-xy^2} \sin x\,dy \right) dx$$

holds for all $r > 0$.

5. (15%) Let $\{f_k\}$ be a sequence of continuous functions of bounded variation on $[a,b]$. Suppose that $f_k \to f$ uniformly on $[a,b]$ and f is of bounded variation. **Prove or disprove** that there exists a constant $M > 0$ such that $V[f_k; a, b] \le M$ for all k, where $V[f; a, b]$ denotes the total variation of f over $[a,b]$.

6. (15%) Let $f(x,y)$, $0 \le x, y \le 1$, satisfy the following conditions: for each x, $f(x,y)$ is an integrable function of y, and there is a $g \in L^1[0,1]$ such that $|(\partial f/\partial x)(x,y)| \le g(y)$ for all $x, y \in [0,1]$. **Prove or disprove** that

$$\frac{d}{dx} \int_0^1 f(x,y)\,dy = \int_0^1 \frac{\partial}{\partial x} f(x,y)\,dy.$$

7. (15%) Prove that, for $1 \le p < \infty$,

$$\left\{ \int \left| \int f(x,y)\,dx \right|^p dy \right\}^{1/p} \le \int \left\{ \int |f(x,y)|^p\,dy \right\}^{1/p} dx.$$

8. (15%) Let g be a measurable function on E. Suppose that there exists a constant $M > 0$ such that $\|gf\|_2 \le M\|f\|_2$ for all $f \in L^2(E)$. Show that $g \in L^\infty(E)$.