國立高雄大學 110 學年度第 1 學期應用數學系博士班資格考試試題

科目：高等線性代數　　　日期：110 年 10 月 15 日　　　考試時間：180 分鐘

Notations:

$\mathbb{M}_n(\mathbb{C})$: the set of $n \times n$ complex matrices.

1. (20%) Prove that $A \in \mathbb{M}_n(\mathbb{C})$ is diagonalizable if and only if there is a positive-definite matrix P such that $P^{-1}AP$ is normal.

2. (10%) Show that any two of the following three properties imply the third:

$$(a) A = A^*; \quad (b) A^* = A^{-1}; \quad (c) A^2 = I.$$

3. (10%) Let A and B be $n \times n$ matrices. Show that for any $n \times n$ matrix X

$$\operatorname{rank}\left(\begin{bmatrix} A & X \\ 0 & B \end{bmatrix}\right) \geq \operatorname{rank}(A) + \operatorname{rank}(B).$$

4. (10%) Let $\mathbb{P}_3$ be the vector space of all real polynomials of degree 3 or less. Define the inner product $< f(x), g(x) >:= \int_0^1 f(x)g(x)dx$. Find an orthonormal basis of $\mathbb{P}_3$.

5. (10%) Let A and B be nonsingular matrices. Find $\begin{bmatrix} A & C \\ 0 & B \end{bmatrix}^{-1}$.

6. (20%) Let V and W be vector spaces, and let $T : V \to W$ be linear. If V is finite dimensional, then $\operatorname{nullity}(T) + \operatorname{rank}(T) = \dim(V)$.

7. (10%) Let $A = \begin{bmatrix} 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 7 \end{bmatrix}$. Find x such that $\|Ax - b\| \leq \|Ay - b\|$ for all $y \in \mathbb{R}^2$.

8. (10%) Let

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} \text{ and } \|A\|_2 = \sup_{x \neq 0} \frac{\|Ax\|_2}{\|x\|_2}.$$

Find $\|A\|_2$.

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