國立高雄大學九十九學年度第一學期應用數學系博士班資格考試試題

科目：高等線性代數　　　　日期: 99 年 9 月 30 日　　　　考試時間：180 分鐘

Answer **all** questions.

Full marks can be obtained by satisfactory answers to all questions.

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1. (15%) Let U, V be subspaces of a finite dimensional vector space and P_3 the space of all real polynomials with degree ≤ 3.

   a) Show that dim (U) + dim(U)= dim $(U+V)$ + dim $(U \cap V)$.

   b) If U is the subspace of P_3 spanned by $1-x+x^2, x-x^2+x^3$ and V the subspace spanned by $1+x, x+x^2, x^2+x^3$, find bases for $U+V$ and $U \cap V$.

2. (10%) Let A and B be $n \times n$ matrices. Suppose $A^2 = A$, $B^2 = B$ and $I - A - B$ is invertible. Show that A and B have the same rank.

3. (10%) Given an infinite system of linear equations

$$x_i + x_{i+2} + x_{i+4} = 0, \quad i = 1, 2, 3 \cdots$$

   with infinite many unknown x_i's. Find all the solutions of the system and determine the number of free parameters required for the solutions.

4. (15%) Given two real symmetric $n \times n$ matrices A and B.

   a) Denoting the maximum and minimum eigenvalues of A by $\lambda_{\max}$ and $\lambda_{\min}$, show that

$$\lambda_{\min} \leq \frac{< x, Ax >}{< x, x >} \leq \lambda_{\max}.$$

   b) If B is positive definite, determine the maximum value of $G(x)$, with $G(x)$ given by

$$G(x) = \frac{< x, Ax >}{< Bx, x >}.$$

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5. (15%) Let A and B be square matrices which are both nilpotent, i.e. $A^i = 0$ and $B^j = 0$ for some positive integers i and j. Prove or provide a counter example for each of the statements below.

   a) $A + B$ is nilpotent.

   b) If $AB = BA$, then $A + B$ is nilpotent.

6. (15%) Let A be a real $m \times n$ matrix and $\|x\|$ denote the Euclidean norm of a vector x. Show that

   a) there exists a unique non-negative $\sigma \in \mathbb{R}$ such that $\|Ax\| \leq \sigma\|x\|$ for all $x \in \mathbb{R}^n$ and $\|Ax\| = \sigma\|x\|$ for some x.

   b) if A is symmetric with $m = n$, then $\sigma = \text{Maximum}\{|\lambda| : \lambda \text{ is an eigenvalue of } A\}$.

7. (10%) Consider a 2×2 matrix

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}.$$

   Show that every real matrix B satisfying $AB = BA$ can be written as $rI + sA$, with $r, s \in \mathbb{R}$ and I being the identity matrix.

8. (10%) Let $A(s)$ be the family of square matrices satisfying the matrix differential equation

$$\frac{dA(s)}{ds} = B\,A(s),$$

   where B is a constant square matrix with the initial condition $A(0) = I$. Find a necessary and sufficient condition of B such that $A(s)$ is orthogonal for all s, that is, $A(s)^T = A(s)^{-1}$ where $A(s)^T$ is the transpose of $A(s)$.