國立高雄大學 109 學年度第 2 學期應用數學系博士班資格考試試題

科目：實變函數論　　日期：110 年 3 月 12 日　　考試時間：180 分鐘

(1) (a) (10 %) A real-valued continuous function f is of bounded variation in $[0,1]$ and absolutely continuous in every interval $[x_0, 1]$, $0 < x_0 < 1$. Prove that f is absolutely continuous in $[0,1]$.

(b) (5 %) Please give an example of continuous function but not absolutely continuous function in $[0,1]$.

(2) (10 %) Let v_n be the volume of the unit ball in $\mathbb{R}^n$. Show by using Fubini's theorem that

$$v_n = 2v_{n-1} \int_0^1 (1-t^2)^{(n-1)/2}\, dt.$$

(3) Let λ be the Lebesgue measure on $\mathbb{R}$.

(a) (10 %) Show that for $f \in L^1([0,\, 2\pi], \lambda)$,

$$\lim_{k\to\infty} \int_0^{2\pi} f(x) \sin kx\, \lambda(dx) = 0.$$

(b) (5 %) Assume that $\{f_k\}$ converges weakly to f in $L^1([0, 2\pi], \lambda)$. Prove or disprove that $f_k \to f$ in $L^1([0, 2\pi], \lambda)$.

(4) (10 %) Let Z be a subset of $\mathbb{R}^1$ with Lebesgue measure zero. Show that the set $\{x^2 : x \in Z\}$ also has measure zero.

(5) (10 %) Let f_k, f be in $L^p(\mathbb{R}^n, \lambda_n)$, $1 < p < \infty$, where λ_n is the Lebesgue measure on $\mathbb{R}^n$. Suppose that $f_k \to f$ a.e. and that $\sup_k \|f_k\|_p \le M < +\infty$. Show that $\int_{\mathbb{R}^n} f_k g\, d\lambda_n \to \int_{\mathbb{R}^n} fg\, d\lambda_n$ for all $g \in L^q(\mathbb{R}^n, \lambda_n)$, $1/p + 1/q = 1$.

(6) (10 %) Let $f_1, \ldots, f_k$ be all Lebesgue measurable functions on $\mathbb{R}$. If $\sum_{i=1}^k 1/p_i = 1/r$, $p_i, r \ge 1$, prove that

$$\|f_1 \cdots f_k\|_r \le \|f_1\|_{p_1} \cdots \|f_k\|_{p_k}.$$

(7) Let $(X, \mathcal{F}, \mu)$ be a measure space.

(a) (10 %) For $\{E_k\} \subset \mathcal{F}$, if $\sum_k \mu(E_k) < +\infty$, show that $\mu(\limsup E_k) = 0$.

(b) (10 %) Suppose that $f \in L^1(X, \mu)$. Prove that to each $\epsilon > 0$ there exists a $\delta > 0$ such that $\int_E |f|\, d\mu < \epsilon$ whenever $\mu(E) < \delta$.

(8) (10 %) Let g be a function in $L^1(\mathbb{R}, \lambda)$ such that $\int_{-\infty}^{\infty} g(x)\, \lambda(dx) = 1$, where λ is the Lebesgue measure. If f is a bounded function on $\mathbb{R}$ and continuous at x_0, prove that

$$\lim_{\epsilon\to 0^+} \frac{1}{\epsilon} \int_{-\infty}^{\infty} f(x) g(\frac{x-x_0}{\epsilon})\, \lambda(dx) = f(x_0).$$