Solution for Problem 01

01. Prove that
$$\pi < \frac{\sin \pi x}{x(1-x)} \le 4, \qquad \text{for } \; 0 < x < 1.$$

Proof. We note that $\dfrac{\sin \pi x}{x(1-x)}$ is symmetric about $x = \dfrac{1}{2}$. It suffices to show that
$$\pi < \frac{\sin \pi x}{x(1-x)} \le 4, \qquad \text{for } \; 0 < x \le \frac{1}{2}. \tag{1}$$

To prove the left inequality of (1), we consider
$$f(x) = \sin \pi x - \pi x(1-x), \qquad 0 \le x \le \frac{1}{2}.$$

Taking derivative, we have $f'(x) = \pi(\cos \pi x - 1 + 2x)$. Define $g(x) = \cos \pi x - 1 + 2x$ for $0 \le x \le \frac{1}{2}$. Then $g''(x) = -\pi^2 \cos \pi x < 0$ for $0 < x < \frac{1}{2}$ and hence the graph of g is concave downward. Since $g(0) = 0 = g(\frac{1}{2})$, we have $g(x) > 0$ for $0 < x < \frac{1}{2}$. Thus, $f'(x) > 0$ for $0 < x < \frac{1}{2}$ and hence f is increasing on $[0, \frac{1}{2}]$. Because $f(0) = 0$, we obtain that $f(x) > 0$ on $(0, \frac{1}{2}]$. Therefore,
$$\pi < \frac{\sin \pi x}{x(1-x)}.$$

To prove the right inequality of (1), we consider
$$h(x) = \sin \pi x - 4x(1-x), \qquad \text{for } \; 0 \le x \le \frac{1}{2}.$$

Differentiating twice produces the following.
$$\begin{aligned} h'(x) &= \pi\Big(\cos \pi x - \frac{4}{\pi} + \frac{8}{\pi}x\Big), \\ h''(x) &= \pi^2\Big(-\sin \pi x + \frac{8}{\pi^2}\Big). \end{aligned}$$

We observe that $\sin \pi x$ is strictly increasing on $[0, \frac{1}{2}]$, $\sin(\pi \cdot 0) = 0 < \frac{8}{\pi^2}$, and $\sin(\pi \cdot \frac{1}{2}) = 1 > \frac{8}{\pi^2}$. Thus, there exists a unique $x_0 \in (0, \frac{1}{2})$ such that $h''(x_0) = 0$ and h'' changes sign once from positive to negative. We notice that
$$\begin{cases} h(0) = 0, \\ h'(0) = \pi\Big(1 - \dfrac{4}{\pi}\Big) < 0, \\ h'' > 0 \qquad \text{on } \; [0, x_0), \end{cases} \qquad \text{and} \qquad \begin{cases} h\Big(\dfrac{1}{2}\Big) = 0, \\ h'\Big(\dfrac{1}{2}\Big) = \pi\Big(\cos \dfrac{\pi}{2} - \dfrac{4}{\pi} + \dfrac{8}{\pi} \cdot \dfrac{1}{2}\Big) = 0, \\ h'' < 0 \qquad \text{on } \; (x_0, \frac{1}{2}]. \end{cases} \tag{2}$$

Then there exists $\delta > 0$ such that

$$h(x) < 0 \qquad \text{on } \ (0, \delta) \text{ and } \left(\frac{1}{2} - \delta, \frac{1}{2}\right). \tag{3}$$

Since h' is increasing on $[0, x_0]$ and is decreasing on $[x_0, \frac{1}{2}]$, $h'(\frac{1}{2}) = 0$, h has only one critical point on $(0, \frac{1}{2})$. By (2) and (3), we get that h has a local minimum and no local maximum on $(0, \frac{1}{2})$, which also implies that $h(x) < 0$ for $0 < x < \frac{1}{2}$. Thus,

$$\frac{\sin \pi x}{x(1-x)} \le 4 \qquad \text{for } \ 0 < x \le \frac{1}{2}.$$

$\square$