Solution to Problem 02.

02. Let $X = \{1, 2, \cdots, n\}$. An antichain $\mathcal{A}$ of X is a collection of subsets of X with the property that no subset in $\mathcal{A}$ is contained in another. For example, $\mathcal{A} = \{\{1, 2\}, \{1, 3\}, \{2, 3, 4\}\}$ is an antichain of $\{1, 2, 3, 4\}$.

a. Determine the maximum size of an antichain of X.

b. Find all antichains of maximum size. Explain your answer.

Proof of a. Define a maximal chain of X to be a chain $A_0 = \emptyset \subset A_1 \subset A_2 \subset \cdots \subset A_n = X$, where $A_i \subseteq X$ and $|A_i| = i$ for $i = 0, 1, 2, \cdots, n$. The number of maximal chains of X is $n!$.

Let $\mathcal{A}$ be an antichain. We count in two different ways the number N of ordered pairs (A, C) such that $A \in \mathcal{A}$ and C is a maximal chain containing A:

(1) Since each maximal chain contains at most one subset in the antichain $\mathcal{A}$,

$$N \leq n!.$$

(2) If $|A| = k$, there are exactly $k!(n-k)!$ maximal chains containing A. Let a_k be the number of k-subsets in the antichain $\mathcal{A}$. Then

$$N = \sum_{k=0}^{n} a_k k!(n-k)!.$$

We calculate that

$$\begin{aligned}
\sum_{k=0}^{n} a_k k!(n-k)! &\leq n! \\
\sum_{k=0}^{n} a_k \frac{k!(n-k)!}{n!} &\leq 1 \\
\sum_{k=0}^{n} \frac{a_k}{\binom{n}{\lfloor n/2 \rfloor}} \leq \sum_{k=0}^{n} \frac{a_k}{\binom{n}{k}} &\leq 1.
\end{aligned}$$

We get that $|\mathcal{A}| = \displaystyle\sum_{k=0}^{n} a_k \leq \binom{n}{\lfloor n/2 \rfloor}$. Furthermore, the collection of all $\lfloor n/2 \rfloor$-subsets of X is an antichain of size $\binom{n}{\lfloor n/2 \rfloor}$. □

Proof of b. If $|\mathcal{A}| = \displaystyle\sum_{k=0}^{n} a_k = \binom{n}{\lfloor n/2 \rfloor}$, then the above inequalities imply that

$$\sum_{k=0}^{n} \frac{a_k}{\binom{n}{\lfloor n/2 \rfloor}} = \sum_{k=0}^{n} \frac{a_k}{\binom{n}{k}};$$

hence, $a_k = 0$ whenever $\binom{n}{k} < \binom{n}{\lfloor n/2 \rfloor}$. Therefore, if n is even, there is only one maximum antichain, that is the collection of all $n/2$-subsets; if n is odd, a maximum antichain consists of some $\lfloor n/2 \rfloor$-subsets and some $\lceil n/2 \rceil$-subsets.

Suppose now that n is odd and $\mathcal{A}$ is an antichain of size $\binom{n}{\lfloor n/2 \rfloor}$, that is, $a_{\lfloor n/2 \rfloor} + a_{\lceil n/2 \rceil} = \binom{n}{\lfloor n/2 \rfloor}$. We have equality at each step, so every maximal chain contains a member in $\mathcal{A}$.

Our aim is to prove that $\mathcal{A}$ consists of all $\lfloor n/2 \rfloor$-subsets or all $\lceil n/2 \rceil$-subsets. Suppose that $\mathcal{A}$ contains some but not all of the $\lceil n/2 \rceil$-subsets. Then we can find sets E and F such that $|E| = |F| = \lceil n/2 \rceil$, $E \in \mathcal{A}$ and $F \notin \mathcal{A}$. We can suppose that $E = \{x_1, \cdots, x_{\lceil n/2 \rceil}\}$ and $F = \{x_i, \cdots, x_{\lceil n/2 \rceil + i - 1}\}$. Since $E \in \mathcal{A}$ and $F \notin \mathcal{A}$, we can find $E^* = \{x_j, \cdots, x_{\lceil n/2 \rceil + j - 1}\} \in \mathcal{A}$ and $F^* = \{x_{j+1}, \cdots, x_{\lceil n/2 \rceil + j}\} \notin \mathcal{A}$ for some $j < i$. Now $E^* \cap F^* \notin \mathcal{A}$ since $E^* \in \mathcal{A}$. Furthermore, $(E^* \cap F^*) \subset F^*$ is part of some maximal chain. By the previous argument, such a chain contains a member in $\mathcal{A}$, implying $F^* \in \mathcal{A}$, a contradiction. Therefore, when n is odd, the only antichains of maximum size are the antichain of all $\lfloor n/2 \rfloor$-subsets of X and the antichain of all $\lceil n/2 \rceil$-subsets of X. $\square$