Solution for Problem 01

Suppose that $f(x)$, $g(x)$, and $h(x)$ are continuous on $[a, b]$ and are differentiable on (a, b). Prove that there exists $\xi \in (a, b)$ such that

$$\begin{vmatrix} f'(\xi) & g'(\xi) & h'(\xi) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0.$$

Proof. Let

$$L(t) = \begin{vmatrix} f(t) - f(a) & g(t) - g(a) & h(t) - h(a) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix}.$$

Then L is continuous on $[a, b]$ and is differentiable on (a, b). Besides, we have $L(a) = L(b) = 0$. By Rolle's Theorem, there exists $\xi \in (a, b)$ such that

$$L'(\xi) = \begin{vmatrix} f'(\xi) & g'(\xi) & h'(\xi) \\ f(a) & g(a) & h(a) \\ f(b) & g(b) & h(b) \end{vmatrix} = 0.$$

$\square$