# Matrix Theory

1 Suppose that $A \in M_{n\times n}(\mathbb{R})$ and $u \in \mathbb{R}^n$. Let $B = \begin{bmatrix} 1 & u^T \\ u & A \end{bmatrix}$. Show that roots of the equation

$$(1-\mu) + u^T(\mu I - A)^{-1}u = 0$$

are eigenvalues of $B$.

Sol. Suppose that $\mu$ is a solution of $(1-\mu)+u^T(\mu I - A)^{-1}u = 0$. Then $\mu I - A$ is invertible. Since

$$\begin{bmatrix} 1-\mu & u^T \\ u & A-\mu I \end{bmatrix} = \begin{bmatrix} 1 & u^T(A-\mu I)^{-1} \\ 0 & I \end{bmatrix} \begin{bmatrix} (1-\mu) - u^T(A-\mu I)^{-1}u & 0 \\ u & A-\mu I \end{bmatrix},$$

we have

$$\begin{aligned}
\det(B-\mu I) &= \det\left(\begin{bmatrix} 1-\mu & u^T \\ u & A-\mu I \end{bmatrix}\right) \\
&= \det\left(\begin{bmatrix} 1 & u^T(A-\mu I)^{-1} \\ 0 & I \end{bmatrix}\right) \det\left(\begin{bmatrix} (1-\mu) - u^T(A-\mu I)^{-1}u & 0 \\ u & A-\mu I \end{bmatrix}\right) \\
&= \det\big((1-\mu) - u^T(A-\mu I)^{-1}u\big)\det(A-\mu I) \\
&= \det\big((1-\mu) + u^T(\mu I - A)^{-1}u\big)\det(A-\mu I)
\end{aligned}$$

Using the fact that $\mu$ is a solution of $(1-\mu)+u^T(\mu I - A)^{-1}u = 0$, we obtain that $\det(B-\mu I) = 0$ and hence, $\mu$ is an eigenvalue of $B$.