Solution of the 2nd problem on April 2015

Fix a value k, $0 \le k < n$, and consider the strategy that rejects the first k prizes and then accepts the first one that is better than all of those first k.

Let $P_k(\text{best})$ denote the probability that the best prize is selected when this strategy is used. Further let X denote the the position of the best prize. This gives

$$P_k(\text{best}) = \sum_{i=1}^{n} P_k(\text{best} \mid X = i) P(X = i) = \frac{1}{n} \sum_{i=1}^{n} P_k(\text{best} \mid X = i).$$

If the overall best prize is among the first k, then no prize is ever selected under the above strategy. That is,

$$P_k(\text{best} \mid X = i) = 0, \text{ if } i \le k.$$

On the other hand, if the best prize is in position i, where $i > k$, then the best prize will be selected if the best of the first $i-1$ prizes is among the first k (for then none of the prizes in positions $k+1, k+2, \ldots, i-1$ would be selected). But conditional on the best prize being in position i, it is easy to verify that all possible orderings of the other prizes remain equally likely, which implies that each of the first $i-1$ prizes is equally likely to be the best of that batch. Hence we see that

$$P_k(\text{best} \mid X = i) = P(\text{best of first } i-1 \text{ is among the first } k \mid X = i) = \frac{k}{i-1}, \text{ if } i > k.$$

From the preceding, we obtain that

$$P_k(\text{best}) = \frac{k}{n} \sum_{i=k+1}^{n} \frac{1}{i-1} \approx \frac{k}{n} \log\left(\frac{n}{k}\right).$$

Now, if we consider the function

$$g(x) = \frac{x}{n} \log\left(\frac{n}{x}\right),$$

then

$$g'(x) = 0 \;\Rightarrow\; x = \frac{n}{e}.$$

Thus, since $P_k(\text{best}) \approx g(k)$, we see that the best strategy of the type considered is to let the first n/e prizes go by and then accept the first one to appear that is better than all of those. In addition, since $g(n/e) = 1/e$, the probability that this strategy selects the best prize is approximately $1/e \approx 0.368$.