A conformal approach for Surface Inpainting Based on Image Inpainting Algorithms

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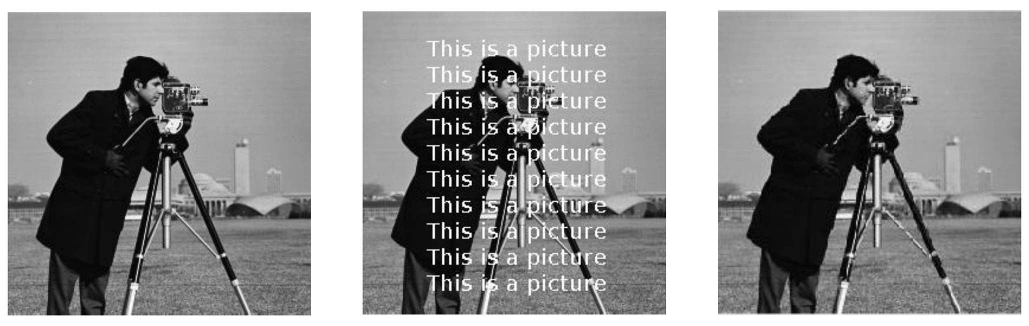
Abstract

In this poster, we study a surface inpainting problem and the missing zones of an incomplete surface can be inpainted by using conformal geometry [2]. Given a Riemann surface S, one can compute its important geometric quantities conformal factor λ and mean curvature H. On the other hand, we can also use λ and H to reconstruct the associated Riemann surface. Based on this idea, the surface inpainting problem can be regarded as the image inpainting problem of λ and H. We numerically study the algorithms for image inpainting models, Li-Yao model and Low Dimensional Manifold Model (LDMM). Li and Yao have presented the mathematical model based on the minimization of an energy function [1]. Numerical schemes are introduced to solve the corresponding Euler-Lagrange equation. As for LDMM [3], it is based on minimization of the dimension of the patch manifold and the Euler-Lagrange equation is obtained by applying the weighted nonlocal Laplacian (WNLL). Moreover, the numerical experiments are presented for recovering the missing region of λ and H by using Li-Yao model and LDMM, respectively. Then the inpainted surface can be reconstructed depending on recovered conformal factor and mean curvature.

Introduction

Surface can be inpainted by using conventional way, but the recovered surface cannot follow the **surface geometry**. According to Riemann surface theory, the surface can be determined by two important geometric quantities, its **conformal factor** λ and **mean curvature** H, such that the surface inpainting problem can be regards as image inpainting problem of λ and H. Image inpainting technique is the image processing skill for recovering missing or unwanted parts in damaged images. One application of image inpainting is removing superimposed text from image in Figure 1.

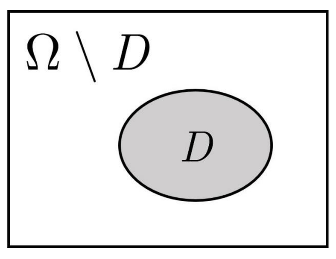


(a) original image (b) damaged image (c) recovered image

Figure 1 : The inpainting result

Notation

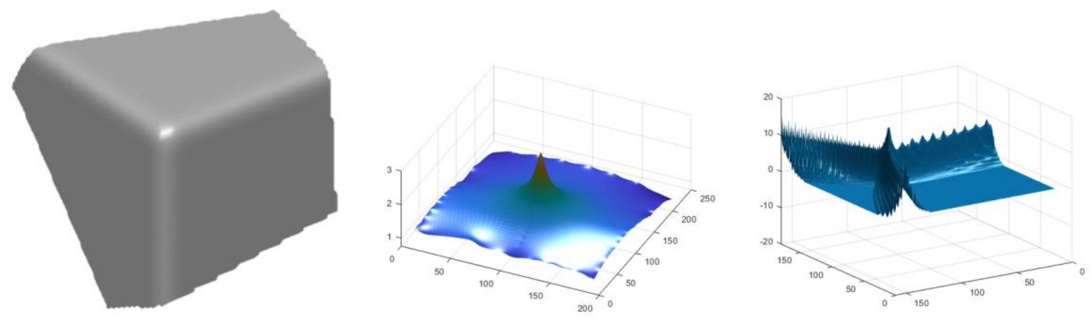
$\mathrm{u} = \mathcal{A}\mathrm{u}_0$, $\mathrm{u}_0 \in \mathbb{R}^{m\times n}$ and $\mathcal{A}$ is the subsample operator



u_0 : original image
u : observed image
Ω : whole domain
D : inpainting domain

Surface inpainting

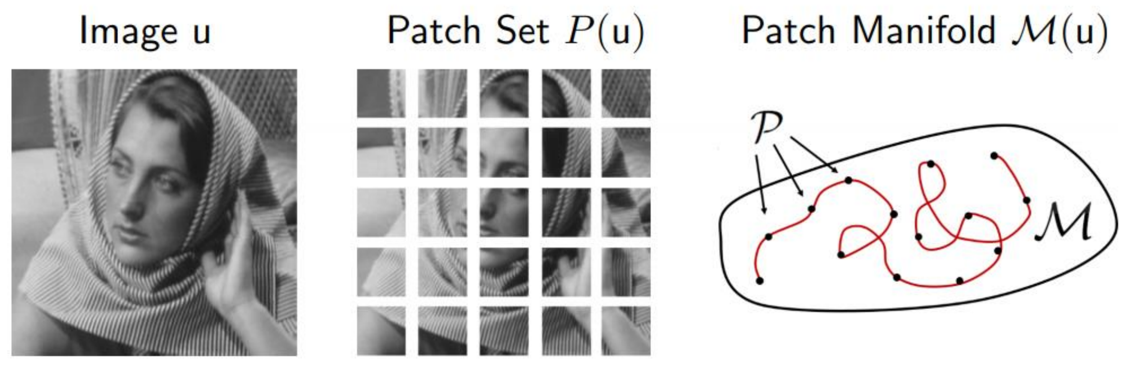
Given a Riemann surface S, one can compute two important geometric quantities, λ and H. On the other hand, we can also use λ and H to reconstruct the associated Riemann surface S.



(a) cube surface S (b) conformal factor λ (c) mean curvature H

Figure 2 : Surface Representation

Inpainting model: LDMM



The optimization problem can be formulated as

$$\min_{\mathrm{u}\in\mathbb{R}^{m\times n}} \sum_{i=1}^{d} ||\nabla_{\mathcal{M}}\alpha_i||^2_{L^2(\mathcal{M})}$$

subject to
$$\alpha_i(\mathcal{P}(\mathrm{u})(x)) = (\mathcal{P}_i\mathrm{u})(x), x\in\Omega, i=1,\dots,d$$
$$\mathrm{u}(x) = \mathrm{u}_0(x), x\in\Omega\setminus D = D^c$$

The Euler-Lagrange equation is obtained by applying WNLL :

$$\sum_{i=1}^{d}\mathcal{P}_i^*\left(\sum_{y\in\Omega}2\bar{w}(x,y)((\mathcal{P}_i\mathrm{u})(x)-(\mathcal{P}_i\mathrm{u})(y))\right)$$
$$+\beta\sum_{i=1}^{d}\mathcal{P}_i^*\left(\sum_{y\in D_i^c}\bar{w}(y,x)((\mathcal{P}_i\mathrm{u})(x)-(\mathcal{P}_i\mathrm{u})(y))\right)=0, x\in D$$

with the constraint $\mathrm{u}(x)=\mathrm{u}_0(x), x\in D^c$

where $\bar{w}(x,y)=w((\bar{\mathcal{P}}\mathrm{u})(x),(\bar{\mathcal{P}}\mathrm{u})(y))$, $w(x,y)=\exp\left(-\frac{\|x-y\|^2}{\sigma(x)^2}\right)$

$D_i^c=\{x\in\Omega:(\mathcal{P}_i\mathrm{u})(x)\text{ is sampled}\}$, $\beta=(\frac{mn}{|D^c|}-1)$, and $\mathcal{P}_i^*=\mathcal{P}_i^{-1}$

Algorithm : LDMM

Input: observed image $\mathrm{u}^{(0)}$
Output: restored image $\mathrm{u}^{(k)}$
$k=0$;
$\gamma^{(0)}=10$;
while *not converge* **do**
1. Extract the semi-local patch set $\bar{\mathcal{P}}(\mathrm{u}^{(k)})$ with $\gamma^{(k)}$;
2. Compute the weight function
$\bar{w}^{(k)}(x,y)=w((\bar{\mathcal{P}}\mathrm{u}^k(x)),(\bar{\mathcal{P}}\mathrm{u}^k(y)))$, $x,y\in\Omega$;
3. Update $\mathrm{u}^{(k+1)}$;
4. Update $\gamma^{(k+1)}=\max(\gamma^{(k)}-1,3)$;
$k\leftarrow k+1$;

end

Inpainting model: Li-Yao model

Li-Yao Model

a simple energy minimization scheme :

$$J_1(\mathrm{u})=\int_\Omega|\nabla\mathrm{u}|\log(1+|\nabla\mathrm{u}|)dx,\qquad \mathrm{u}|_{\Omega\setminus D}=\mathrm{u}_0|_{\Omega\setminus D}$$

The steepest descent equation for $J_1(\mathrm{u})$ is given by

$$\begin{cases}\mathrm{u}_t=\nabla\cdot\left\{\left[\frac{\log(1+|\nabla\mathrm{u}|)}{|\nabla\mathrm{u}|}+\frac{1}{1+|\nabla\mathrm{u}|}\right]\nabla\mathrm{u}\right\}, & \text{in } D\\ \mathrm{u}=\mathrm{u}_0, & \text{on } \partial D\\ \mathrm{u}|_{t=0}=\mathrm{u}_0, & \text{in } \overline{D}\end{cases}$$

Algorithm 1: Li-Yao Model

Input: observed image $\mathrm{u}^{(0)}$
Output: restored image $\mathrm{u}^{(k)}$
$k=0$;
while *not converge* **do**
1. Compute $F(\mathrm{u}^{(k)})=\frac{\log(1+|\nabla\mathrm{u}^{(k)}|)}{|\nabla\mathrm{u}^{(k)}|}+\frac{1}{1+|\nabla\mathrm{u}^{(k)}|}$;
2. Approximate $\mathrm{u}^{(k+1)}\approx\mathrm{u}^{(k)}+dt\{\nabla\cdot[F(\mathrm{u}^{(k)})\nabla\mathrm{u}^{(k+1)}]\}$;
$k\leftarrow k+1$;

end

Inpainting model: GD method

In addition to Li-Yao model and LDMM, we apply other inpainting model which is called GD method

$$\mathrm{u}'=\arg\min_{\mathrm{u}}\int_{\Omega\setminus D}|\Delta\mathrm{u}|^2$$

under the constraint $\mathrm{u}'|_{\Omega\setminus D}=\mathrm{u}|_{\Omega\setminus D}$

where u' is the final results after iterations

Let $z=u+iv$, $\frac{\partial}{\partial z}=\frac{1}{2}(\frac{\partial}{\partial u}-i\frac{\partial}{\partial v})$ and $\frac{\partial}{\partial\bar{z}}=\frac{1}{2}(\frac{\partial}{\partial u}+i\frac{\partial}{\partial v})$

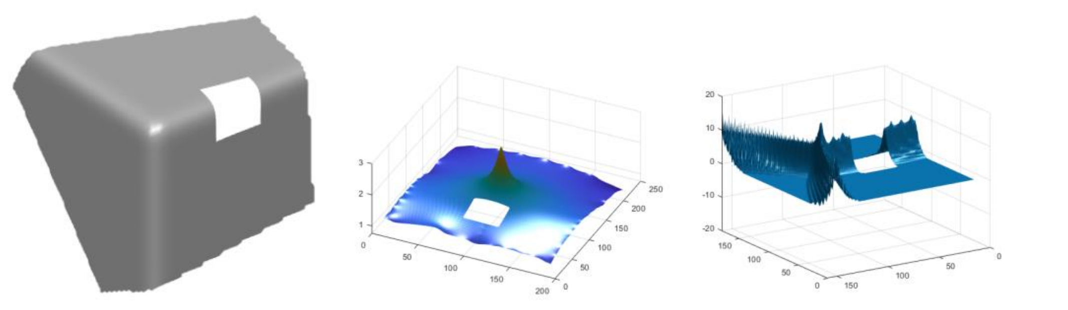
With the conformal parameterization $\phi(u,v)$,

we have $\langle\phi_z,\phi_z\rangle=\langle\phi_{\bar{z}},\phi_{\bar{z}}\rangle=\frac{\lambda^2}{2}$ and $\langle\phi_z,\phi_{\bar{z}}\rangle=0$

Consider $\mu=\langle\vec{n},\phi_{zz}\rangle$,

the natural frame $(\phi_z,\phi_{\bar{z}},\vec{n})$ satisfies

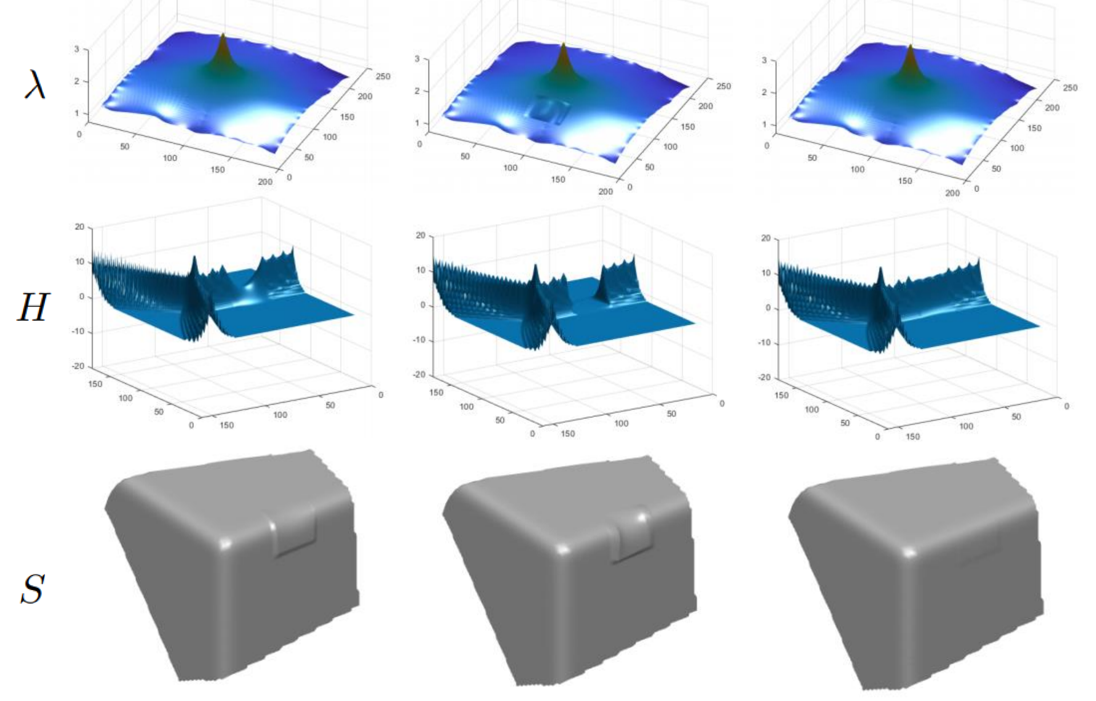
$$\left[\begin{array}{l}\frac{\partial}{\partial z}\phi_z=\frac{2}{\lambda}\lambda_z\phi_z+\mu\vec{n}\\ \frac{\partial}{\partial z}\phi_{\bar{z}}=\frac{\lambda^2}{2}H\vec{n}\\ \frac{\partial}{\partial z}\vec{n}=-H\phi_z-\frac{2\mu\phi_{\bar{z}}}{\lambda^2}\end{array}\right.$$



(a) incomplete surface (b) incomplete λ (c) incomplete H

Figure 3 : The surface contains a hole that needs to be filled

Results and discussion



(a) GD method (b) Li-Yao model (c) LDMM

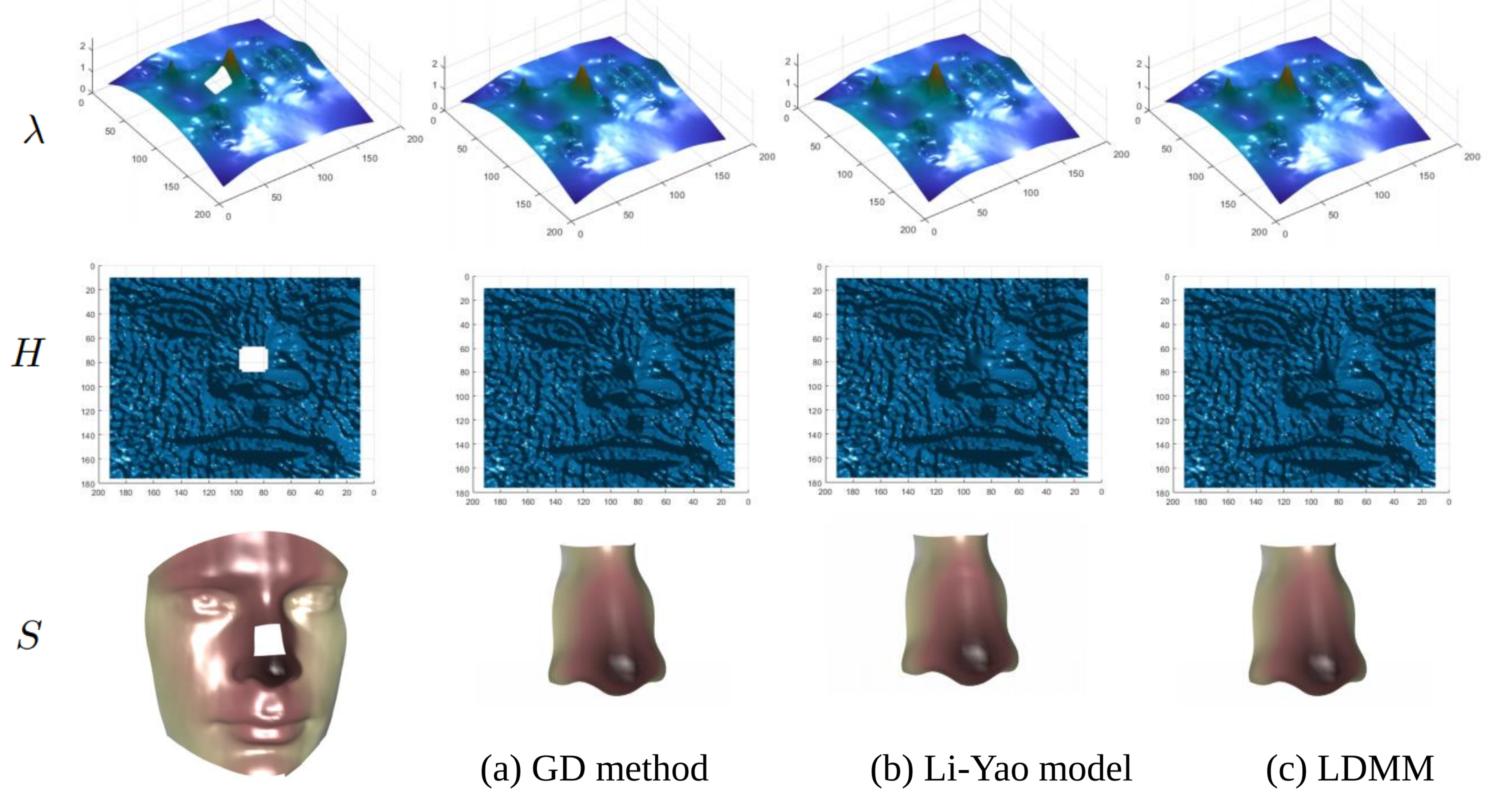
Figure 4 : Surface inpainting of the cube surface

	GD method	Li-Yao model	LDMM
mean error	3.686e-03	8.567e-03	4.942e-04
max error	5.818e-02	7.815e-02	8.366e-03
sd error	5.281e-03	9.339e-03	7.517e-04

Table 1 : Error between original and inpainted cube surfaces

	GD method	Li-Yao model	LDMM
mean error	4.328e-04	5.877e-04	4.995e-04
max error	2.217e-03	2.327e-03	3.145e-03
sd error	4.258e-04	5.204e-04	5.259e-04

Table 2 : Error between original and inpainted human face surfaces



(a) GD method (b) Li-Yao model (c) LDMM

Figure 5 : Surface inpainting of the human face surface

In Figure 4, it is obviously that the main reason is that the mean curvature does not recover well so that the reconstructions do not look good (a,b). In contrast, using LDMM achieves satisfying result that looks better on fitting the boundary of the surface (c).

In Figure 5, we show zoom-in views of the recovered surface. Table 2 shows the error between the original and inpainted human face surface. The reconstructed surface are all recovered well, especially in GD method and LDMM.

Conclusions

- The surface inpainting problem can be regards as the image inpainting problem of the conformal factor and mean curvature.
- We recover the conformal factor and mean curvature by using inpainting models, and surface can be reconstructed depending on inpaitned conformal factor and mean curvature.
- By comparing of GD method, Li-Yao model and LDMM, LDMM shows a better performance in convergence speed and inpainting quality.

References

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