國立高雄大學理學院微積分基礎能力會考 104–106

試題整理

第 P 章（函數）題目

(P-1) Find a such that $f(x) = \cos \pi x + \sqrt{a - x^2}$ has its domain $[-5, 5]$.

解答

(P-1) **Ans.** $a = 25$.

The domain of $f(x) = \cos \pi x + \sqrt{a - x^2}$ is defined for x such that $f(x)$ exists. Hence

$$\begin{aligned} \text{Domain of } f &= \{x : a - x^2 \geq 0\} \\ &= \begin{cases} \varnothing & \text{if } a < 0, \\ [-\sqrt{a}, \sqrt{a}] & \text{if } a \geq 0. \end{cases} \end{aligned}$$

Hence domain of f is equal to $[-5, 5]$ if and only if $\sqrt{a} = 5$ or $a = 25$.

第一章（極限與連續）題目

(1-1) What is the limit of $f(x) = 4$ as x approaches π?

(1-2) What is the limit of $f(x) = 2x + 1$ as x approaches 0?

(1-3) Find the limit of $\lim\limits_{x\to 2} \dfrac{\sqrt{x^2+12}-4}{x-2}$.

(1-4) Find the limit of $\lim\limits_{x\to 1} \left(\dfrac{1}{x-1} + \dfrac{1}{x^2-3x+2}\right)$.

(1-5) Find the limit of $\lim\limits_{x\to 2^-} \dfrac{x^2+x-6}{|x-2|}$.

(1-6) Let $f(x) = \dfrac{x^2-3}{2x-4}$. Answer the following questions:

(A) $\lim\limits_{x\to\infty} f(x)$; (B) $\lim\limits_{x\to 2^-} f(x)$; (C) $\lim\limits_{x\to 2^+} f(x)$. (D) Is the line $y = \frac{1}{2}(x+1)$ a slant asymptote of the graph of f.

(1-7) Evaluate $\lim\limits_{h\to 0} \frac{1-\cos h}{h^2}$.

(1-8) Which of the following statements are not true?

(A) $\lim\limits_{x\to\pi} \frac{\sin x}{x} = 0$; (B) $\lim\limits_{x\to\infty} \frac{\sin x}{x} = 1$; (C) $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$; (D) $\lim\limits_{x\to -\infty} \frac{\sin x}{x} = 0$;

(1-9) Find the limit of $\lim\limits_{x\to 0} \frac{\sin(\sin x)}{x}$.

(1-10) What is the limit of $f(x) = \frac{x}{|x|}$ as x approaches 0 from the left?

(1-11) Find the limit of $\lim\limits_{x\to -6^+} (-3\lfloor |x| \rfloor - 8)$, where $f(x) = \lfloor x \rfloor$ represents the greatest integer which is smaller or equal to x.

(1-12) Find the limit of $\lim\limits_{x\to 0^+} \sqrt{x}\cos^2 \frac{1}{x}$.

(1-13) Consider the function

$$f(x) = \begin{cases} \frac{x^2-x}{x^2-1} & x < 1, \\ \frac{1}{2} & x = 1, \\ \frac{x^3-x}{x^2-1} & x > 1, \end{cases}$$

Then at point $x = 1$, is f continuous? continuous from the left? continuous from the right?

(1-14) A business has a cost of $C = 1.7x + 700$ for producing x units. The average cost per unit is $\bar{C} = \frac{C}{x}$. Find the limit of $\bar{C}$ as x approaches infinity.

(1-15) Suppose that f and g are continuous functions with $f(5) = 5$ and $\lim\limits_{x\to 5}(2f(x) - g(x)) = 6$. Find $g(5)$.

(1-16) Which one of the following functions is not continuous on $\mathbb{R}$?

(A) $f(x) = x^2+x+1$; (B) $f(x) = \dfrac{1}{x}$; (C) $f(x) = \dfrac{1}{x^2+1}$; (D) none of above.

(1-17) Is it true that if f is continuous at a, then so is $|f|$?

(1-18) Find all the x-values at which $f(x) = \frac{\cos x^2}{x^2-2x}$ is not continuous.

(1-19) Find the value of c such that $f(x) = \begin{cases} 4-x^2 & x \le c, \\ x & x > c, \end{cases}$ is continuous on $\mathbb{R}$.

(1-20) For what values of x is

$$f(x) = \begin{cases} \frac{x^2-x-6}{x-3} & x \ne 3, \\ 5 & x = 3, \end{cases}$$

continuous?

(1-21) For what values of x is $f(x) = \begin{cases} 0 & x \in \mathbb{Q}, \\ x & x \in \mathbb{Q}^c, \end{cases}$ continuous?

(1-22) For what values of x is $f(x) = \begin{cases} 4x & x \in \mathbb{Q}, \\ x^2+3 & x \in \mathbb{Q}^c, \end{cases}$ continuous?

(1-23) Is it true that if $x = a$ is a vertical asymptote of $y = f(x)$, then $f(a)$ exists.

(1-24) Let $f(x) = -\dfrac{8}{x^2-4}$. Does $f(x)$ have vertical asymptotes? How many, if any?

(1-25) Find the vertical asymptotes (if any) of the function $f(x) = \tan(15x)$.

(1-26) Find the horizontal asymptotes (if any) of the function $f(x) = \sqrt{x^6+5x^3} - x^3$, $x \ge 0$.

(1-27) Consider $f(x) = 7x + \cos x$. Then $f(x)$ has a slant asymptote $y = 7x$?

解答

(1-1) **Ans.** 4.

Since $f(x) = 4$ is a constant function, it is continuous on $\mathbb{R}$. So $\lim\limits_{x\to\pi} f(x) = f(\pi) = 4$.

(1-2) **Ans.** 1.

Since $f(x) = 2x+1$ is a polynomial function, it is continuous on $\mathbb{R}$. So $\lim\limits_{x\to 0} f(x) = f(0) = 1$.

(1-3) **Ans.** $\frac{1}{2}$.

$$\begin{aligned}\lim_{x\to 2}\frac{\sqrt{x^2+12}-4}{x-2} &= \lim_{x\to 2}\frac{(\sqrt{x^2+12}-4)(\sqrt{x^2+12}+4)}{(x-2)(\sqrt{x^2+12}+4)}\\ &= \lim_{x\to 2}\frac{(\sqrt{x^2+12})^2-4^2}{(x-2)(\sqrt{x^2+12}+4)}\\ &= \lim_{x\to 2}\frac{x^2-4}{(x-2)(\sqrt{x^2+12}+4)} = \lim_{x\to 2}\frac{(x-2)(x+2)}{(x-2)(\sqrt{x^2+12}+4)}\\ &= \lim_{x\to 2}\frac{x+2}{\sqrt{x^2+12}+4} = \frac{2+2}{\sqrt{2^2+12}+4} = \frac{1}{2}.\end{aligned}$$

(1-4) **Ans.** -1.

$$\begin{aligned}\lim_{x\to 1}\left(\frac{1}{x-1}+\frac{1}{x^2-3x+2}\right) &= \lim_{x\to 1}\left(\frac{1}{x-1}+\frac{1}{(x-1)(x-2)}\right)\\ &= \lim_{x\to 1}\frac{(x-2)+1}{(x-1)(x-2)} = \lim_{x\to 1}\frac{1}{x-2} = \frac{1}{1-2} = -1.\end{aligned}$$

(1-5) **Ans.** -5.

Since $|x-2| = -(x-2)$ whenever $x<2$, we have that

$$\begin{aligned}\lim_{x\to 2^-}\frac{x^2+x-6}{|x-2|} &= \lim_{x\to 2^-} -\frac{x^2+x-6}{(x-2)} = \lim_{x\to 2^-} -\frac{(x-2)(x+3)}{(x-2)}\\ &= \lim_{x\to 2^-} -(x+3) = -(2+3) = -5.\end{aligned}$$

(1-6) **Ans.** (A) $+\infty$; (B) $-\infty$; (C) $+\infty$; (D) No.

(A) $\lim\limits_{x\to\infty}\frac{x^2-3}{2x-4} = \lim\limits_{x\to\infty}\frac{x-\frac{3}{x}}{2-\frac{4}{x}} = +\infty$ ($\because \lim\limits_{x\to\infty}(x-\frac{3}{x}) = +\infty$ and $\lim\limits_{x\to\infty}(2-\frac{4}{x}) = 2$.)

(B) Since $\lim\limits_{x\to 2}(x^2-3) = 1$ and $\lim\limits_{x\to 2}(2x-4) = 0$, we have that $\lim\limits_{x\to 2} f(x) = \lim\limits_{x\to 2}\frac{x^2-3}{2x-4}$ does not exist. Moreover, since $2x-4<0$ whenever $x<2$, we conclude that $\lim\limits_{x\to 2^-} f(x) = -\infty$.

(C) By similar argument as above and since $2x-4>0$ whenever $x>2$, we conclude that $\lim\limits_{x\to 2^+} f(x) = +\infty$.

(D) We compute that

$$f(x)-\frac{1}{2}(x+1) = \frac{x^2-3}{2x-4}-\frac{1}{2}(x+1) = \frac{x-1}{2x-4}.$$

Hence, $\lim\limits_{x\to\pm\infty}\left[f(x)-\frac{1}{2}(x+1)\right] = \frac{1}{2}\neq 0$. It implies $y=\frac{1}{2}(x+1)$ is *not* a slant asymptote of f.

In fact, to find a slant asymptote of f, we compute that

$$f(x)-(ax+b) = \frac{(-2a+1)x^2+2(2a-b)x+(4b-3)}{2x-4}.$$

Then it can be found that $\lim\limits_{x\to\pm\infty}[f(x)-(ax+b)] = 0$ if and only if $-2a+1=0$ and $2a-b=0$. Equivalently, $a=1/2$ and $b=1$. Consequently, $y=\frac{1}{2}x+1$ is a slant asymptote of f.

(1-7) **Ans.** $\frac{1}{2}$.

Note that

$$\begin{aligned}&\cos(2\theta) = 1 - 2\sin^2(\theta)\\ &\Leftrightarrow 1 - \cos(2\theta) = 2\sin^2(\theta)\\ &\Leftrightarrow 1 - \cos(h) = 2\sin^2(h/2) \quad \text{(by letting } h = 2\theta).\end{aligned}$$

So

$$\begin{aligned}\lim_{h\to 0}\frac{1-\cos h}{h^2} &= \lim_{h\to 0}\frac{2\sin^2(h/2)}{h^2}\\ &= \lim_{x\to 0}\frac{2\sin^2(x)}{(2x)^2} \quad \text{(by letting } x = h/2)\\ &= \lim_{x\to 0}\frac{1}{2}\left(\frac{\sin(x)}{x}\right)^2\\ &= \frac{1}{2}\left(\lim_{x\to 0}\frac{\sin(x)}{x}\right)^2 = \frac{1}{2}\cdot 1^2 = \frac{1}{2}.\end{aligned}$$

(1-8) **Ans.** (B) is false.

(A) $\lim\limits_{x\to\pi}\frac{\sin x}{x} = \frac{\sin\pi}{\pi} = 0$;

(B) Let $y = 1/x$. Then $x \to \infty$ is equivalent to $y \to 0^+$. So $\lim\limits_{x\to\infty}\frac{\sin x}{x} = \lim\limits_{y\to 0^+} y\sin\frac{1}{y} = 0$ by Squeeze Theorem;

(C) $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$;

(D) Let $y = 1/x$. Then $x \to -\infty$ is equivalent to $y \to 0^-$. So $\lim\limits_{x\to-\infty}\frac{\sin x}{x} = \lim\limits_{y\to 0^-} y\sin\frac{1}{y} = 0$ by Squeeze Theorem.

(1-9) **Ans.** 1.

$\lim\limits_{x\to 0}\frac{\sin(\sin x)}{x} = \lim\limits_{x\to 0}\left[\frac{\sin(\sin x)}{\sin x}\frac{\sin x}{x}\right]$. Note that $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$. On the other hand, let $y = \sin x$. Then since $x \to 0$ implies that $y \to 0$, we have that $\lim\limits_{x\to 0}\frac{\sin(\sin x)}{\sin x} = \lim\limits_{y\to 0}\frac{\sin(y)}{y} = 1$. Hence, $\lim\limits_{x\to 0}\frac{\sin(\sin x)}{x} = \lim\limits_{x\to 0}\left[\frac{\sin(\sin x)}{\sin x}\frac{\sin x}{x}\right] = \left[\lim\limits_{x\to 0}\frac{\sin(\sin x)}{\sin x}\right]\left[\lim\limits_{x\to 0}\frac{\sin x}{x}\right] = 1$.

(1-10) **Ans.** -1.

Since when x is negative, $f(x) = \frac{x}{|x|} = \frac{x}{-x} = -1$, we have that $\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0^-} -1 = -1$.

(1-11) **Ans.** -23.

Since, for $x \in (-6,-5)$, $|x| \in (5,6)$ and hence $-3\lfloor|x|\rfloor - 8 = -3\cdot 5 - 8 = -23$. It implies that $\lim\limits_{x\to -6^+}(-3\lfloor|x|\rfloor - 8) = -23$.

(1-12) **Ans.** 0. Because $0 \le \sqrt{x}\cos^2\frac{1}{x} < \sqrt{x}$ for all $x > 0$, and $\lim\limits_{x\to 0} 0 = \lim\limits_{x\to 0^+}\sqrt{x} = 0$, we have that $\lim\limits_{x\to 0^+}\sqrt{x}\cos^2\frac{1}{x} = 0$ by Squeeze Theorem.

(1-13) **Ans.** At $x = 1$, $f(x)$ is only continuous from the left but neither continuous from the right nor continuous.

By direct computation,

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \frac{x^2 - x}{x^2 - 1} = \lim_{x\to 1^-} \frac{x(x-1)}{(x+1)(x-1)} = \lim_{x\to 1^-} \frac{x}{(x+1)} = \frac{1}{2},$$
$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} \frac{x^3 - x}{x^2 - 1} = \lim_{x\to 1^+} \frac{x(x^2-1)}{x^2 - 1} = \lim_{x\to 1^+} x = 1.$$

Since $f(1) = 1/2$, we conclude that, at $x = 1$, $f(x)$ is only continuous from the left but neither continuous from the right nor continuous.

(1-14) **Ans.** 1.7.

By direct computation, $\lim\limits_{x\to\infty} \bar{C} = \lim\limits_{x\to\infty} \frac{1.7x+700}{x} = 1.7$.

(1-15) **Ans.** 4.

Since f is a continuous function with $f(5) = 5$, $\lim\limits_{x\to 5} f(x) = f(5) = 5$. Moreover, by the condition, $\lim\limits_{x\to 5} [2f(x) - g(x)] = 6$, we have that

$$\begin{aligned}\lim_{x\to 5} g(x) &= \lim_{x\to 5} \{2f(x) - [2f(x) - g(x)]\} \\ &= 2\lim_{x\to 5} f(x) - \lim_{x\to 5} [2f(x) - g(x)] = 10 - 6 = 4.\end{aligned}$$

It follows that $g(5) = \lim\limits_{x\to 5} g(x) = 4$ since g is a continuous function.

(1-16) **Ans.** (B).

(A) $f(x) = x^2 + x + 1$ is continuous on $\mathbb{R}$ since f is a polynomial;

(B) $f(x) = \dfrac{1}{x}$ is Not continuous on $\mathbb{R}$ since $\lim\limits_{x\to 0^+} = \infty$;

(C) $f(x) = \dfrac{1}{x^2+1}$ is continuous on $\mathbb{R}$ since $x^2 + 1 \geq 1 > 0$ for all $x \in \mathbb{R}$.

(1-17) **Ans.** It is true.

Since $g(x) := |x|$ is a continuous function on $\mathbb{R}$, and, by condition, f is continuous at a, we get that $|f|(x) = (g \circ f)(x)$ is also continuous at a.

(1-18) **Ans.** $x = 0, 2$.

The possible points for $f(x) = \frac{\cos x^2}{x^2 - 2x}$ at which f is not continuous must satisfy $x^2 - 2x = 0$, i.e., $x = 0, 2$. Moreover, since both $\cos(0) \neq 0$ and $\cos(4) \neq 0$, we have that $f(x)$ is not continuous at points $x = 0, 2$.

(1-19) **Ans.** $c = \frac{-1\pm\sqrt{17}}{2}$.

Note that both $4 - x^2$ and x are polynomials so they are continuous on $\mathbb{R}$. Hence $f(x)$ defined by

$$f(x) = \begin{cases} 4 - x^2 & x \leq c, \\ x & x > c, \end{cases}$$

must be continuous on $\mathbb{R}-\{c\}$ for any $c \in \mathbb{R}$. To make $f(x)$ is also continuous at c, it suffices to let $f(c)=\lim_{x\to c^-} f(x)=\lim_{x\to c^+} f(x)$. Equivalently, $f(c)=4-c^2=c$ or $c^2+c-4=0$. That is, $c=\frac{-1\pm\sqrt{17}}{2}$.

(1-20) **Ans.** $\mathbb{R}$.
Since $\frac{x^2-x-6}{x-3}$ is a rational function and its denominator is nonzero for $x \neq 3$, we have that $\frac{x^2-x-6}{x-3}$ is continuous on $\mathbb{R}-\{3\}$. Hence function f defined by

$$f(x)=\begin{cases} \frac{x^2-x-6}{x-3} & x\neq 3, \\ 5 & x=3, \end{cases}$$

is continuous on $\mathbb{R}-\{3\}$. Moreover, at $x=3$, since $f(3)=5$ and, by direct computation, $\lim_{x\to 3} f(x)=\lim_{x\to 3}\frac{x^2-x-6}{x-3}=\lim_{x\to 3}\frac{(x-3)(x+2)}{x-3}=\lim_{x\to 3}(x+2)=5$, we obtain that $f(3)=\lim_{x\to 3} f(x)$. So $f(x)$ is continuous at $x=3$. In summary, f is continuous on $\mathbb{R}$.

(1-21) **Ans.** $x=0$.

Case I: We first show that, the limit $\lim_{x\to 0} f(x)=0$. It is easy to see that $-|x|\le f(x)\le |x|$. Then, by Squeeze Theorem, $\lim_{x\to 0} f(x)=0$ since $\lim_{x\to 0} -|x|=\lim_{x\to 0}|x|=0$.

Case II: We now show that, for any $p\neq 0$, the limit $\lim_{x\to p} f(x)$ does not exist.

(反證法) Suppose that the statement is false. That is, $\lim_{x\to p} f(x)=L$ for some $L\in\mathbb{R}$. Then we first note that, by definition of the limit,

$$\forall \varepsilon>0,\ \exists\,\delta>0 \text{ s.t. } |f(x)-L|<\varepsilon, \quad \text{whenever } 0<|x-p|<\delta. \tag{1}$$

Take $\varepsilon=\frac{|p|}{3}>0$. Then, by (1), we get that

$$\exists\,\delta_1>0 \text{ s.t. } |f(x)-L|<\frac{|p|}{3}, \quad \text{whenever } 0<|x-p|<\delta_1.$$

Define

$$\delta_2=\min\{\delta_1,\frac{|p|}{5}\}. \tag{2}$$

Then

$$|f(x)-L|<\frac{|p|}{3}, \quad \text{whenever } 0<|x-p|<\delta_2. \tag{3}$$

Choose arbitrary rational number r and irrational number s satisfying

$$0<|r-p|,|s-p|<\delta_2. \tag{4}$$

Then, by (3),

$$|f(r)-L|<\frac{|p|}{3} \text{ and } |f(s)-L|<\frac{|p|}{3}.$$

It follows, by triangular inequality, that

$$\begin{aligned} &|f(r)-f(s)| \le |f(r)-L| + |L-f(s)| < \frac{|p|}{3} + \frac{|p|}{3} = \frac{2}{3}|p| \\ &\Rightarrow |0-s| = |s| < \frac{2}{3}|p| \end{aligned} \tag{5}$$

by the definition of f. Consequently,

$$\begin{aligned} |p| &= |s+(s-p)| \le |s| + |s-p| \\ &\le \frac{2}{3}|p| + \delta_2 \quad \text{(by (5) and (4))} \\ &\le \frac{2}{3}|p| + \frac{|p|}{5} \quad \text{(by (2))} \\ &= \frac{13}{15}|p|, \end{aligned}$$

a contradiction. Hence, the statement "for any $p \neq 0$, the limit $\lim_{x\to p} f(x)$ does not exist" is true.

(1-22) **Ans.** $x = 1, 3$.
We compute that $4x = x^2 + 3$ if and only if $x^2 - 4x + 3 = (x-1)(x-3) = 0$, equivalently, $x = 1, 3$. Hence $f(x)$ is continuous at $x = 1, 3$. Remark that to prove this question is somewhat difficult. Here we give a simple explanation: The limit $\lim_{x\to c} f(x)$ exists $\Leftrightarrow \lim_{x\in\mathbb{Q}\to c} f(x) = \lim_{x\in\mathbb{Q}^c\to c} f(x)$.

(1-23) **Ans.** No.
For instance, for $f(x) = \frac{1}{x}$, $x = 0$ is a vertical asymptote of $y = f(x)$. However, $f(a)$ does not exist.

(1-24) **Ans.** Two ($x = -2$ and $x = 2$).
Since $f(x) = -\frac{8}{x^2-4}$ is a rational function with positive numerator, we have that f has a vertical asymptote at point p if and only if the denominator of $f(p)$, i.e., $p^2 - 4 = 0$. That is, $p = \pm 2$. Consequently, $f(x)$ has two vertical asymptotes, i.e., $x = -2$ and $x = 2$.

(1-25) **Ans.** $x = c$, where $c \in \{\frac{\pi}{30} + \frac{n\pi}{15} : n \in \mathbb{Z}\}$.
Since function $f(x) = \tan(x)$ has vertical asymptotes at $x \in \{\frac{\pi}{2} + n\pi : n \in \mathbb{Z}\}$. Hence the vertical asymptotes of the function $f(x) = \tan(15x)$ occur at $x \in \{\frac{\pi}{30} + \frac{n\pi}{15} : n \in \mathbb{Z}\}$. Hence the vertical asymptotes of the function $f(x) = \tan(15x)$ are $x = c$, where $c \in \{\frac{\pi}{30} + \frac{n\pi}{15} : n \in \mathbb{Z}\}$

(1-26) **Ans.** $y = \frac{5}{2}$.

We compute that

$$\begin{aligned}
\lim_{x\to\pm\infty} f(x) &= \lim_{x\to\pm\infty}\left[\sqrt{x^6+5x^3}-x^3\right] \\
&= \lim_{x\to\pm\infty}\left[\frac{(\sqrt{x^6+5x^3}-x^3)(\sqrt{x^6+5x^3}+x^3)}{\sqrt{x^6+5x^3}+x^3}\right] \\
&= \lim_{x\to\pm\infty}\left[\frac{5x^3}{\sqrt{x^6+5x^3}+x^3}\right] \\
&= \lim_{x\to\pm\infty}\left[\frac{5}{\sqrt{1+5\frac{1}{x^3}}+1}\right] = \frac{5}{2}.
\end{aligned}$$

Hence the horizontal asymptote of the function $f(x) = \sqrt{x^6+5x^3}-x^3$ is $y = \frac{5}{2}$.

(1-27) **Ans.** No.

We compute that $\lim_{x\to\pm\infty}[f(x)-(7x)] = \lim_{x\to\pm\infty}\cos(x)$, which does not exist. Hence, $y = 7x$ is not a slant asymptote of $f(x) = 7x + \cos x$.

第二章（微分）題目

(2-1) Is it true that if f is continuous at a, then f is differentiable at a?

(2-2) Find the limit of $\lim_{x\to\frac{\pi}{4}} \dfrac{\tan x - 1}{x - \frac{\pi}{4}}$.

(2-3) Find the limit of $\lim_{x\to 1} \frac{x^{2016}-1}{x-1}$.

(2-4) Find the derivative of $f(x) = \sqrt{10}$.

(2-5) Find the derivative of $f(x) = \frac{1}{x^4}$.

(2-6) Find the derivative of $f(x) = \sqrt{\dfrac{x-1}{x+1}}$.

(2-7) Find the derivative of $f(x) = \cos(2x^4 - 6)$.

(2-8) Find the derivative of $f(x) = \sin^2(\pi x - 2)$.

(2-9) Consider $f(x) = \dfrac{g(h(x))}{h(x)}$, where $g(x)$ and $h(x)$ satisfy $g(3) = 3$, $g'(3) = 7$, $h(6) = 3$, $h'(6) = -2$. Find $f'(6)$.

(2-10) For what values of a and b will

$$f(x) = \begin{cases} ax + b & x \le -1, \\ ax^3 + x + 2b & x > -1, \end{cases}$$

be differentiable for all values of x?

(2-11) If $f(x) = \sqrt{x}g(x)$, where $g(4) = 8$ and $g'(4) = 7$, find $f'(4)$.

(2-12) Find $f'(x)$ if it is known that $\frac{d}{dx} f(2x) = x^2$.

(2-13) Find $\dfrac{d^{2016}}{dx^{2016}} \cos(x)$.

(2-14) Suppose that $f'(a) = 1$ for some constant a. Then find the possible value a for the limit $\lim_{x\to a} \frac{f(x)-f(a)}{\sqrt[3]{x}-1} = 3$?

(2-15) Find an equation for the line that is tangent to the curve $y = x^3 - x$ at the point $(-1, 0)$.

(2-16) Determine the slope of the graph of $x^2y^2 - 2x = 4 - 4y$ at the point $(2, -2)$.

(2-17) Find the tangent line to the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$ at the point $(2, -\sqrt{3})$.

(2-18) Find the slope of the tangent to $x^3 + xy - \cos(xy) = 0$ at the point $(1, 0)$.

解答

(2-1) **Ans.** No.

Consider $f(x) = |x|$. Then f is continuous at 0 but f is not differentiable at 0.

(2-2) **Ans.** 2.

Since $\tan(\pi/4) = 1$,

$$\lim_{x\to\frac{\pi}{4}} \frac{\tan x - 1}{x - \frac{\pi}{4}} = \lim_{x\to\frac{\pi}{4}} \frac{\tan x - \tan(\pi/4)}{x - \frac{\pi}{4}} = \tan'(\pi/4) = \sec^2(\pi/4) = 2.$$

(2-3) **Ans.** 2016.

It is easy to see that $\lim\limits_{x\to 1} \frac{x^{2016}-1}{x-1} = f'(1)$ for $f(x) := x^{2016}$. Since $f'(x) = 2016x^{2015}$ and $f'(1) = 2016$, we have that $\lim\limits_{x\to 1} \frac{x^{2016}-1}{x-1} = 2016$.

(2-4) **Ans.** 0.

Since $f(x) = \sqrt{10}$ is a constant function, $f'(x) = 0$.

(2-5) **Ans.** $-4x^{-5}$.

Consider $f(x) = x^{-4}$. Then, by direct computation, $f'(x) = -4x^{-5}$.

(2-6) **Ans.** $\dfrac{1}{(x+1)^2}\sqrt{\dfrac{x+1}{x-1}}$.

Consider $f(x) = \sqrt{\dfrac{x-1}{x+1}}$. Then, by the chain rule,

$$\begin{aligned}
\frac{d}{dx}\sqrt{\frac{x-1}{x+1}} &= \left[\frac{d}{dy}\sqrt{y}\right]_{y=\frac{x-1}{x+1}} \left[\frac{d}{dx}\frac{x-1}{x+1}\right] \\
&= \frac{1}{2}\left[\frac{1}{\sqrt{y}}\right]_{y=\frac{x-1}{x+1}} \frac{(x+1)-(x-1)}{(x+1)^2} \\
&= \frac{1}{2}\sqrt{\frac{x+1}{x-1}}\frac{2}{(x+1)^2} \\
&= \frac{1}{(x+1)^2}\sqrt{\frac{x+1}{x-1}}.
\end{aligned}$$

(2-7) **Ans.** $-(8x^3)\sin(2x^4-6)$.

Consider $f(x) = \cos(2x^4 - 6)$. Then, by the chain rule,

$$\begin{aligned}
\frac{d}{dx}\cos(2x^4-6) &= \left[\frac{d}{dy}\cos y\right]_{y=2x^4-6} \left[\frac{d}{dx}(2x^4-6)\right] \\
&= \left[-\sin(y)\right]_{y=2x^4-6} \left[8x^3\right] \\
&= -(8x^3)\sin(2x^4-6).
\end{aligned}$$

(2-8) **Ans.** $2\pi\sin(\pi x-2)\cos(\pi x-2)$.

Consider $f(x)=\sin^2(\pi x-2)$. Then, by the chain rule,

$$\begin{aligned}\frac{d}{dx}\sin^2(\pi x-2)&=\left[\frac{d}{dz}z^2\right]_{z=\sin(\pi x-2)}\left[\frac{d}{dy}\sin(y)\right]_{y=\pi x-2}\left[\frac{d}{dx}(\pi x-2)\right]\\&=[2z]_{z=\sin(\pi x-2)}[\cos(y)]_{y=\pi x-2}[(\pi)]\\&=2\sin(\pi x-2)\cos(\pi x-2)(\pi)\\&=2\pi\sin(\pi x-2)\cos(\pi x-2).\end{aligned}$$

(2-9) **Ans.** -4.

Consider $f(x)=\dfrac{g(h(x))}{h(x)}$. Then, we compute that

$$\begin{aligned}\frac{d}{dx}f(x)&=\frac{d}{dx}\frac{g(h(x))}{h(x)}\\&=\frac{\left[\frac{d}{dx}g(h(x))\right]h(x)-g(h(x))h'(x)}{[h(x)]^2}\\&=\frac{[g'(h(x))h'(x)]\,h(x)-g(h(x))h'(x)}{[h(x)]^2}.\end{aligned}$$

Then, since $g(3)=3$, $g'(3)=7$, $h(6)=3$, $h'(6)=-2$, we have that

$$\begin{aligned}f'(6)&=\frac{[g'(h(6))h'(6)]\,h(6)-g(h(6))h'(6)}{[h(6)]^2}\\&=\frac{[g'(3)h'(6)]\cdot 3-g(3)h'(6)}{[3]^2}\\&=\frac{[(7)(-2)]\cdot 3-3\cdot(-2)}{[3]^2}=\frac{-14+2}{3}=-4.\end{aligned}$$

(2-10) **Ans.** $a=-1/2$ and $b=1$.

Consider

$$f(x)=\begin{cases}ax+b & x\le -1,\\ ax^3+x+2b & x>-1.\end{cases}$$

Note first that, since $f(x)$ is of polynomial form on intervals $(-\infty,-1)$ and $(-1,\infty)$, we get that f is differentiable for $x\in\mathbb{R}-\{-1\}$. Secondly, to have f be also differentiable at -1, it suffices to show that the chosen (a,b) can make the following two assertions (a)–(b) hold true.

(a) f is continuous at 1, i.e, $\lim_{x\to-1^-}f(x)=\lim_{x\to-1^+}f(x)$.

(b) f is differentiable at 1, i.e, $\lim_{x\to-1^-}\frac{f(x)-f(-1)}{x-(-1)}=\lim_{x\to-1^+}\frac{f(x)-f(-1)}{x-(-1)}$.

In fact, we compute that $\lim_{x\to-1^-}f(x)=-a+b$, $\lim_{x\to-1^+}f(x)=-a-1+2b$, $\lim_{x\to-1^-}\frac{f(x)-f(-1)}{x-(-1)}=a$ and $\lim_{x\to-1^+}\frac{f(x)-f(-1)}{x-(-1)}=3a+1$. Then, to have (a)-(b), we solve that

$$\begin{cases}-a+b=-a-1+2b\\ a=3a+1\end{cases}$$

Then we find that $a = -1/2$ and $b = 1$.

(2-11) **Ans.** 16.

Let $f(x) = \sqrt{x}g(x)$. Then, by direct computation, $f'(x) = \frac{1}{2\sqrt{x}}g(x) + \sqrt{x}g'(x)$. Then, since $g(4) = 8$ and $g'(4) = 7$, we have that $f'(4) = \frac{1}{2\sqrt{4}}g(4) + \sqrt{4}g'(4) = \frac{1}{4} \cdot 8 + 2 \cdot 7 = 16$.

(2-12) **Ans.** $f'(x) = x^2/8$.

Note that, by the chain rule, $\frac{d}{dx}f(2x) = 2f'(2x)$. Then, by the condition, $\frac{d}{dx}f(2x) = x^2$, we have that $2f'(2x) = x^2$. It implies that $f'(2x) = x^2/2$ and hence $f'(X) = (X/2)^2/2 = X^2/8$ by letting $X = 2x$. Equivalently, $f'(x) = x^2/8$.

(2-13) **Ans.** $\cos(x)$.

Note that

$$\begin{aligned}
\frac{d}{dx}\cos(x) &= -\sin(x), \\
\frac{d^2}{dx^2}\cos(x) &= -\cos(x), \\
\frac{d^3}{dx^3}\cos(x) &= \sin(x), \\
\frac{d^4}{dx^4}\cos(x) &= \cos(x), \\
&\vdots
\end{aligned}$$

It is then easy to see that $\frac{d^n}{dx^n}\cos(x) = \cos(x)$ for all $\{n = 4m : m \in \mathbb{N}\}$. Hence $\dfrac{d^{2016}}{dx^{2016}}\cos(x) = \cos(x)$.

(2-14) **Ans.** $a = 1$.

Note that

$$\begin{aligned}
\lim_{x\to a}\frac{f(x)-f(a)}{\sqrt[3]{x}-1} &= \lim_{x\to a}\left[\frac{f(x)-f(a)}{x-a}\,\frac{x-a}{\sqrt[3]{x}-1}\right] \\
&= f'(a)\lim_{x\to a}\frac{x-a}{\sqrt[3]{x}-1} = \lim_{x\to a}\frac{x-a}{\sqrt[3]{x}-1} \\
&= \begin{cases} \frac{a-a}{\sqrt[3]{a}-1} & \text{if } \sqrt[3]{a} \neq 1 \\ \lim\limits_{x\to a}\frac{x-1}{\sqrt[3]{x}-1} & \text{if } \sqrt[3]{a} = 1 \end{cases} \\
&= \begin{cases} 0 & \text{if } a \neq 1, \\ \lim\limits_{x\to a}\frac{x-a}{\sqrt[3]{x}-1} & \text{if } a = 1. \end{cases}
\end{aligned}$$

Hence, the possible value a for the limit $\lim\limits_{x\to a}\frac{f(x)-f(a)}{\sqrt[3]{x}-1} = 3$ is 1. In fact, form above, we compute that, for $a=1$,

$$
\begin{aligned}
\lim_{x\to a}\frac{f(x)-f(a)}{\sqrt[3]{x}-1} &= \lim_{x\to 1}\frac{x-1}{\sqrt[3]{x}-1}\\
&= \lim_{x\to 1}\frac{(x-1)\left[(\sqrt[3]{x})^2+(\sqrt[3]{x})+1\right]}{(\sqrt[3]{x}-1)\left[(\sqrt[3]{x})^2+(\sqrt[3]{x})+1\right]}\\
&= \lim_{x\to 1}\frac{(x-1)\left[(\sqrt[3]{x})^2+(\sqrt[3]{x})+1\right]}{(\sqrt[3]{x})^3-1}\\
&= \lim_{x\to 1}\left[(\sqrt[3]{x})^2+(\sqrt[3]{x})+1\right]=3.
\end{aligned}
$$

(2-15) **Ans.** $y=2(x+1)$.

By definition, the equation of the tangent line of curve $y=x^3-x$ at the point $(-1,0)$ is

$$
\begin{aligned}
& y-0=\left.\frac{dy}{dx}\right|_{(x,y)=(-1,0)}[x-(-1)]\\
\Leftrightarrow\ & y=(3x^2-1)_{(x,y)=(-1,0)}(x+1)\\
\Leftrightarrow\ & y=2(x+1).
\end{aligned}
$$

(2-16) **Ans.** $\frac{7}{6}$.

From the equation $x^2y^2-2x=4-4y$, we compute that

$$
\begin{aligned}
& (2x)y^2+x^2(2yy')-2=-4y'\\
\Leftrightarrow\ & (2x^2y+4)y'=2-(2x)y^2\\
\Leftrightarrow\ & y'=\frac{1-xy^2}{x^2y+2}
\end{aligned}
$$

if $x^2y+2\neq 0$. Hence, the slope of the graph of $x^2y^2-2x=4-4y$ at the point $(2,-2)$ is $\frac{1-2\cdot(-2)^2}{2^2\cdot(-2)+2}=\frac{7}{6}$.

(2-17) **Ans.** $y=\frac{\sqrt{3}(x-8)}{6}$.

From the equation $\frac{x^2}{16}+\frac{y^2}{4}=1$, we compute that

$$
\begin{aligned}
& \frac{2x}{16}+\frac{2yy'}{4}=0\\
\Leftrightarrow\ & x+4yy'=0\\
\Leftrightarrow\ & y'=\frac{-x}{4y}
\end{aligned}
$$

if $y\neq 0$. Hence the tangent line to the ellipse $\frac{x^2}{16}+\frac{y^2}{4}=1$ at the point $(2,-\sqrt{3})$ is $y-(-\sqrt{3})=\frac{-2}{4\cdot(-\sqrt{3})}(x-2)$ or, equivalently $y=\frac{-2}{4\cdot(-\sqrt{3})}(x-2)-\sqrt{3}=\frac{\sqrt{3}(x-8)}{6}$.

(2-18) **Ans.** -3.

From the equation $x^3+xy-\cos(xy)=0$, we compute that

$$3x^2+y+xy'-\sin(xy)\cdot(y+xy')=0$$

Hence at point $(1,0)$, we have that the slop (y') satisfies $3+y'-\sin(0)\cdot(0+y')=0$. Equivalently, $y'=-3$.

(3-1) Find all critical points of $f(x) = x^{1/3}(x-4)$.

(3-2) Find all critical points of $f(x) = \sin|x|$.

(3-3) Consider the function $f(x) = x^3 - 3x^2 + 1$, $-\frac{1}{2} \le x < 4$. How many critical points do f have? Does f have an absolute maximum or an absolute minimum?

(3-4) Find the absolute maximum value of the function $f(x) = x\sqrt{9-x^2}$, $-3 \le x \le 3$.

(3-5) Find the intervals where the function $f(x) = x\sqrt{30-x^2}$ is increasing.

(3-6) Consider $f(x) = x^4 - 6x^2 + 1$.

- (A) Find the intervals such that f increasing;
- (B) Find the intervals such that f decreasing;
- (C) Find all the inflection points of f;
- (D) Find the absolute minimum value of f.

(3-7) How many real roots does the equation $4x^5 + x^3 + 2x + 4 = 0$ have?

(3-8) How many solutions does the equation $x^5 - 7x + c = 0$ have in the interval $[-1, 1]$?

(3-9) Consider $f(x) = 7x + \cos x$. Which of the following statements is true?

- (A) $f(x)$ has infinitely many roots.
- (B) $f(x)$ is a periodic function.
- (C) $f(x)$ has exactly one root.

(3-10) Which one of the following statement is false?

- (A) If $f'(x)$ exists and is nonzero for all x, then $f(1) \neq f(0)$.
- (B) If f is even, then f' is even.
- (C) If f is increasing and $f(x) > 0$ on an interval I, then $g(x) = 1/f(x)$ is decreasing on I.

(3-11) Find the point on the line $y = 4x + 8$ that is closest to the origin.

(3-12) How many points of inflection does the function $f(x) = x^6 - 15x^2 + 1$ have?

(3-13) Which one of the following functions has no inflection point at $x = 0$?
(A) $f(x) = x^{1/3}$; (B) $f(x) = x^{5/3}$; (C) $f(x) = x^4$; (D) $f(x) = \sin x$;

(3-14) The graph of the function $f(x) = \cos x$ on the interval $(\pi/2, \pi)$ is
(A) increasing; (B) decreasing; (C) concave upward; (D) concave downward.

(3-15) The graph of the function $f(x) = x^4 - 4x^3 + 10$ on the interval $(2, 3)$ is
(A) increasing; (B) decreasing; (C) concave upward; (D) concave downward.

(3-16) Find the differential dy of $y = \sqrt{x}$ at $x = 4$ with $dx = 3$.

(3-17) Find the differential dy of $y = x\cos(7x)$.

(3-18) Use the linear approximation of the function $f(x) = \sqrt{9 - x}$ to approximate $\sqrt{9.09}$.

(3-19) Consider

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ x\sin(\frac{1}{x}) & \text{if } x \neq 0. \end{cases}$$

Which of the following statements are true?

(A) $f(x)$ is differentiable at $x = 0$;

(B) $f(x)$ is continuous at $x = 0$;

(C) $f(x)$ has a global maximum value;

(D) $f(x)$ has an inflection point.

(3-20) Consider

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ x\sin(\frac{1}{x}) & \text{if } x \neq 0. \end{cases}$$

Which one of the following statements is false?

(A) The Intermediate Value Theorem can be applied to f on the interval $[-1, 1]$;

(B) The Mean Value Theorem can be applied to f on the interval $[-1, 1]$;

(C) $-|x| \leq f(x) \leq |x|$.

解答

(3-1) **Ans.** 0, 1.
We compute that $f'(x) = \frac{1}{3}x^{-2/3}(x-4) + x^{1/3} = \frac{1}{3}x^{-2/3}(x-4+3x) = \frac{4}{3}x^{-2/3}(x-1)$. Hence, the critical points of $f(x)$ are 0 and 1.

(3-2) **Ans.** 0, $x = \frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$.

By the definition of $f(x) = \sin(|x|)$,

- As $x > 0$, $f(x) = \sin(x)$ and hence $f'(x) = \cos(x)$. So $f'(x) = 0$ if and only if $x = \frac{\pi}{2} + n\pi$, $n \in \mathbb{N} \cup \{0\}$.
- As $x < 0$, $f(x) = \sin(-x)$ and hence $f'(x) = -\cos(-x)$. So $f'(x) = 0$ if and only if $x = -\frac{\pi}{2} - n\pi$, $n \in \mathbb{N} \cup \{0\}$.
- At $x = 0$, we compute that

$$\begin{aligned} f'(0) &= \lim_{x\to 0} \frac{\sin(|x|) - \sin(0)}{x - 0} \\ &= \lim_{x\to 0} \frac{\sin(|x|)}{x} \\ &= \begin{cases} \lim\limits_{x\to 0^+} \frac{\sin(x)}{x} = 1 \\ \lim\limits_{x\to 0^-} \frac{\sin(-x)}{x} = \lim\limits_{x\to 0^-} \frac{-\sin(x)}{x} = -1 \end{cases} \end{aligned}$$

  It means the right-hand and the left-hand derivative of $f(x)$ are not equal. So $f'(0)$ does not exist.

Hence the critical points are $x = 0$, $x = \frac{\pi}{2} + n\pi$, $n \in \mathbb{Z}$.

(3-3) **Ans.** 2 critical points; f has the global minimum at $x = 2$; f has no global maximum.

- By the definition, $f(x) = x^3 - 3x^2 + 1$, $-\frac{1}{2} \le x < 4$. We then compute that $f'(x) = 3x^2 - 6x = 3x(x-2)$. Hence the critical points of f are 0 and 2. Note that both 0 and 2 lies in $[-\frac{1}{2}, 4)$.
- We compute that $f''(x) = 6(x-1)$, $f''(0) = -6 < 0$ and $f''(2) = 6 > 0$. Hence f has an <u>local maximum</u> at 0 and an <u>local minimum</u> at 2.
- We compute that $f(0) = 1$, $f(2) = -3$, $f(-\frac{1}{2}) = \frac{1}{8}$ and $\lim\limits_{x\to 4^-} f(x) = 17$. It follows that f has <u>global minimum at $x = 2$</u> but <u>no global maximum</u> (since $\lim\limits_{x\to 4^-} f(x) > f(0) > \cdots$).

(3-4) **Ans.** $\frac{9}{2}$.

Let $f(x) = x\sqrt{9 - x^2}$, $-3 \le x \le 3$. Then we compute that

$$\begin{aligned} f'(x) &= \sqrt{9 - x^2} + x\frac{-2x}{2\sqrt{9 - x^2}} \\ &= \sqrt{9 - x^2} - \frac{x^2}{\sqrt{9 - x^2}} \\ &= \frac{9 - 2x^2}{\sqrt{9 - x^2}}. \end{aligned}$$

Hence, the critical points of f are $x = \pm\frac{3}{\sqrt{2}}$. Next, we compute $f(-3) = f(3) = 0$, $f(-\frac{3}{\sqrt{2}}) = -\frac{9}{2}$, $f(\frac{3}{\sqrt{2}}) = \frac{9}{2}$. It implies that the maximum value of f is $\frac{9}{2}$, which occurs at $\frac{3}{\sqrt{2}}$.

(3-5) **Ans.** $(-\sqrt{15}, \sqrt{15})$.

Let $f(x) = x\sqrt{30 - x^2}$. Then the domain of f is $[-\sqrt{30}, \sqrt{30}]$. Then we compute that

$$\begin{aligned} f'(x) &= \sqrt{30 - x^2} + x\frac{-2x}{2\sqrt{30 - x^2}} \\ &= \sqrt{30 - x^2} - \frac{x^2}{\sqrt{30 - x^2}} \\ &= \frac{30 - 2x^2}{\sqrt{30 - x^2}}. \end{aligned}$$

So, in $[-\sqrt{30}, \sqrt{30}]$, $f'(x) > 0$ if and only if $x \in [-\sqrt{15}, \sqrt{15}]$. It means f is increasing on $[-\sqrt{15}, \sqrt{15}]$.

(3-6) **Ans.** (A) $[-\sqrt{3}, 0)$ and $[\sqrt{3}, \infty)$. (B) $(-\infty, -\sqrt{3}]$ and $[0, \sqrt{3}]$. (C) ± 1. (D) -8.

Consider $f(x) = x^4 - 6x^2 + 1$. We compute that $f'(x) = 4x^3 - 12x = 4x(x^2 - 3)$. Hence $f'(x) = 0$ if and only if $x = 0, \pm\sqrt{3}$. Moreover,

(a) $f'(x) > 0 \Leftrightarrow x \in (-\sqrt{3}, 0)$ or $(\sqrt{3}, \infty)$. It implies f is increasing on $[-\sqrt{3}, 0)$ and $[\sqrt{3}, \infty)$.

(b) $f'(x) < 0 \Leftrightarrow x \in (-\infty, -\sqrt{3})$ or $(0, \sqrt{3})$. It implies f is increasing on $(-\infty, -\sqrt{3}]$ and $[0, \sqrt{3}]$.

Next, we compute that $f''(x) = 12x^2 - 12 = 12(x^2 - 1)$. So $f''(x) = 0$ if and only if $x = \pm 1$. Moreover, we observe that $f''(x) > 0$ whenever $x \in (-\infty, 1) \cup (1, \infty)$, and $f''(x) < 0$ whenever $x \in (-1, 1)$. Hence ± 1 are the inflection points.

Finally, since $f(0) = 1$, $f(\pm\sqrt{3}) = -8$, $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to-\infty} f(x) = \infty$. It means that the absolute minimum value of f is -8.

(3-7) **Ans.** 1.

Let $f(x) = 4x^5 + x^3 + 2x + 4$. Then we compute $f'(x) = 20x^4 + 3x^2 + 2 > 0$ for all $x \in \mathbb{R}$. It means f is increasing on $\mathbb{R}$. Hence f has at most 1 real root. On the other hand, since $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to\infty} f(x) = \infty$. By the intermediate value theorem, f has at least 1 real root. In conclusion, f has exactly one root.

(3-8) **Ans.** f has at most one root on $[-1, 1]$.

Let $f(x) = x^5 - 7x + c$, where $x \in [-1, 1]$. We compute that $f'(x) = 5x^4 - 7 < 0$ on $[-1, 1]$, which implies that f is decreasing on [-1,1]. So f has at most 1 root on $[-1, 1]$.

More precisely, by the intermediate value theorem and since $f(-1) = 6 + c$ and $f(1) = -6 + c$, $f(x)$ has exactly one root if $f(-1)f(1) \leq 0$ (i.e., $c \in [-6, 6]$). Otherwise, if $c \notin [-6, 6]$, $f(x)$ has no root in $[-1, 1]$.

(3-9) **Ans.** Only (B) is true.

Let $f(x) = 7x + \cos x$. Then we compute that $f'(x) = 7 - \sin(x) \geq 6 > 0$ for all

$x \in \mathbb{R}$. It implies that f is increasing and hence it has at most one root. Moreover, since $\lim_{x\to-\infty} f(x) = -\infty$ and $\lim_{x\to-\infty} f(x) = \infty$, we have that f has at least one root, by the intermediate value theorem. In conclusion, f has exactly one root. Remark that this also implies that f is not a periodic function.

(3-10) **Ans.** Only (B) is false.

(a) (A):"If $f'(x)$ exists and is nonzero for all x, then $f(1) \neq f(0)$." is true. If not, i.e, $f(1) = f(0)$, then, by the MVT, there exists $c \in (0,1)$ such that $f'(c) = 0$.

(b) (B):"If f is even, then f' is even." is false. Note that, by the definition,

$$\begin{aligned} f'(-x) &= \lim_{y\to-x} \frac{f(y)-f(-x)}{y-(-x)} \\ &= \lim_{y\to-x} \frac{f(y)-f(-x)}{y+x} \\ &= \lim_{z\to x} \frac{f(-z)-f(-x)}{-z+x} \quad (\text{let } z=-y) \\ &= \lim_{z\to x} \frac{f(z)-f(x)}{-z+x} \quad (\because f \text{ is even.}) \\ &= \lim_{z\to x} -\frac{f(z)-f(x)}{z-x} \\ &= -f'(x). \end{aligned}$$

So f' is odd, if f is even.

(c) (C):"If f is increasing and $f(x) > 0$ on an interval I, then $g(x) = 1/f(x)$ is decreasing on I." is true. Because, for any $x_1 < x_2$ on I,

$$\begin{aligned} g(x_1) &= 1/f(x_1) \\ &< 1/f(x_2) \quad (\because f \text{ is increasing and positive} \therefore 0 < f(x_1) < f(x_2).) \\ &= g(x_2). \end{aligned}$$

(3-11) **Ans.** $(-\frac{32}{17}, \frac{8}{17})$.

To find the point on the line $y = 4x + 8$ that is closest to the origin, we solve the global minimum value for the following function:

$$\begin{aligned} f(x) &= (x-0)^2 + (y-0)^2 \\ &= x^2 + (4x+8)^2 \\ &= 17x^2 + 64x + 64. \end{aligned}$$

Clearly, the minimum value occurs at $x = -\frac{64}{2\cdot 17} = -\frac{32}{17}$ and $y = -4 \cdot \frac{32}{17} + 8 = \frac{8}{17}$.

(3-12) **Ans.** 2.

To find inflection points for the function $f(x) = x^6 - 15x^2 + 1$. We compute that $f''(x) = 30x^4 - 15 = 15(x^4 - 1) = 15(x-1)(x+1)(x^2+1)$. So $f''(x) = 0$ if and only if $x = \pm 1$. Moreover, since $f''(x) < 0$ whenever $x \in (-1,1)$ and $f''(x) > 0$ whenever $x \in (-\infty, -1)$ or $(1, \infty)$, it follows that $x = \pm 1$ are inflection points.

(3-13) **Ans.** (C).

(a) Let $f(x) = x^{1/3}$. We compute that $f''(x) = -\frac{2}{9}x^{-5/3}$. So $f''(x) > 0$ for $x < 0$ and $f''(x) < 0$ for $x > 0$. It implies that $x = 0$ is an inflection point.

(b) Let $f(x) = x^{5/3}$. We compute that $f''(x) = \frac{10}{9}x^{-1/3}$. So $f''(x) < 0$ for $x < 0$ and $f''(x) > 0$ for $x > 0$. It implies that $x = 0$ is an inflection point.

(c) Let $f(x) = x^4$. We compute that $f''(x) = 12x^2$. So $f''(x) > 0$ for all $x \neq 2$. It implies that $x = 0$ is Not an inflection point.

(d) Let $f(x) = \sin(x)$. We compute that $f''(x) = -\sin x$. So $f''(x) > 0$ for $x < 0$ and $f''(x) < 0$ for $x > 0$. It implies that $x = 0$ is an inflection point.

(3-14) **Ans.** Only (B) and (C) are true.
Let $f(x) = \cos x$ on the interval $(\pi/2, \pi)$. We compute that $f'(x) = -\sin(x) < 0$ on $(\pi/2, \pi)$, which means f is decreasing on $(\pi/2, \pi)$. Next, we compute that $f''(x) = -\cos(x) > 0$ on $(\pi/2, \pi)$, which means f is concave upward on $(\pi/2, \pi)$.

(3-15) **Ans.** Only (B) and (C) are true..
Let $f(x) = x^4 - 4x^3 + 10$ on the interval $(2, 3)$. We compute that $f'(x) = 4x^3 - 12x^2 = 4x^2(x-3) < 0$ on $(2, 3)$, which means f is decreasing on $(2, 3)$. Next, we compute that $f''(x) = 12x^2 - 12 = 12(x^2 - 1) > 0$ on $(2, 3)$, which means f is concave upward on $(\pi/2, \pi)$.

(3-16) **Ans.** $\frac{3}{4}$.
Let $y = \sqrt{x}$. We compute that $\frac{dy}{dx} = \frac{1}{2\sqrt{x}}$. So $dy = \frac{1}{2\sqrt{x}}dx$. Then at $x = 4$ and $dx = 3$, we have $dy = \frac{1}{2\sqrt{4}} \cdot 3 = \frac{3}{4}$.

(3-17) **Ans.** $dy = [\cos(7x) - 7x\sin(7x)]dx$.
Let $y = x\cos(7x)$. We compute that $\frac{dy}{dx} = \cos(7x) - 7x\sin(7x)$. So $dy = [\cos(7x) - 7x\sin(7x)]dx$.

(3-18) **Ans.** 3.015.
Let $f(x) = \sqrt{9-x}$. Then we compute that $f'(x) = \frac{-1}{2\sqrt{9-x}}$. So the linear approximation

$$(f(x) \approx)\, y = f(c) + f'(c)(x - c)$$

for the function $f(x) = \sqrt{9-x}$ at $c = 0$ is

$$(f(x) \approx)\, y = f(0) + f'(0)(x - 0) = 3 - \frac{1}{6}(x - 0).$$

So

$$\sqrt{9.09} = f(-0.09) \approx 3 - \frac{1}{6}(-0.09 - 0) = 3.015.$$

(3-19) **Ans.** Only (B) and (D) are true.

Let

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ x\sin(\frac{1}{x}) & \text{if } x \neq 0. \end{cases}$$

Then

(a) $\lim_{x\to 0} f(x) = \lim_{x\to 0} x\sin(\frac{1}{x}) = 0 = f(0)$, by squeeze theorem. Hence, f is continuous at 0.

(b) $f'(0) = \lim_{x\to 0} \frac{x\sin(\frac{1}{x})}{x} = \lim_{x\to 0} \sin(\frac{1}{x})$: does not exist. Hence, f is Not differentiable at 0.

(c) Note that $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} x\sin(\frac{1}{x}) = \lim_{y\to 0^+} \frac{\sin(y)}{y} = 1$. However, we can prove that $f(x) < 1$ for all $x \in \mathbb{R}$. It follows that f has no global maximum value.

Note that to prove $f(x) < 1$ for all $x \in \mathbb{R}$, it suffices to prove that $x\sin(\frac{1}{x}) < 1$ for all $x > 0$ since $f(x) = f(-x)$ for any $x \neq 0$. Equivalently, it suffices to prove that $\sin(\frac{1}{x}) < \frac{1}{x}$ for $x > 0$ or $\sin(t) < t$ for $t > 0$ (letting $t = 1/x$). Indeed, by the MVT, $\sin(t) - \sin 0 = \cos(c)(t - 0)$, which follows that $|\sin t| < |t|$ and $\sin t < t$ for $t > 0$. Hence, we obtain that $f(x) < 1$ for all $x \in \mathbb{R}$.

(d) We compute that, for $x \neq 0$, $f''(x) = -\frac{\sin(\frac{1}{x})}{x^3}$. Hence $f''(\frac{1}{2\pi}) = 0$ and $f''(x) > 0$ when x is slightly greater than $\frac{1}{2\pi}$ and $f''(x) < 0$ when x is slightly smaller than $\frac{1}{2\pi}$. It implies that $\frac{1}{2\pi}$ is an inflection point.

(3-20) **Ans.** Only (B) is false.

Let

$$f(x) = \begin{cases} 0 & \text{if } x = 0, \\ x\sin(\frac{1}{x}) & \text{if } x \neq 0. \end{cases}$$

Then

(a) Since f is continuous on $[-1, 1]$ (see the answer given in (3-19)), the Intermediate Value Theorem can be applied to f.

(b) Since f is Not differentiable on the whole interval $(-1, 1)$, the Mean Value Theorem can Not be applied to f on the interval $[-1, 1]$;

(c) The inequality is $-|x| \leq f(x) \leq |x|$ clear.

第四章（積分）題目

(4-1) Find the indefinite integral $\int 7x^6 \sin x^7 dx$.

(4-2) Suppose that the graph of f passes through the point $(4, 69)$ and the slope of its tangent line at $(x, f(x))$ is $10x - 4$. Find $f(1)$.

(4-3) Write the limit $\lim\limits_{|\triangle|\to 0} \sum\limits_{i=1}^{n} \frac{6}{c_i^2} \triangle x_i$ as a definite integral $[8, 10]$, on the interval where c_i is any point in the i^{th} subinterval.

(4-4) Express the limit $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{i^4}{n^5}$ as a definite integral.

(4-5) Evaluate the integral $\int_{-1}^{3} |2x - x^2| dx$.

(4-6) Evaluate the integral $\int_1^2 x^3 \sqrt{x^4 + 5}\, dx$.

(4-7) Find the average value of $f(x) = \frac{5(x^2+5)}{x^2}$ on the interval $[1, 3]$.

(4-8) Evaluate the definite integral $\int_0^{\pi} \sec^2(x/4) dx$.

(4-9) Evaluate the definite integral $\int_0^{3\pi/2} |\sin(x)| dx$.

(4-10) Evaluate the definite integral $\int_{-\pi/2}^{\pi/2} \sin^2(x) \cos(x) dx$.

(4-11) Is it true that $\int_a^b f(x + h) dx = \int_{a+h}^{b+h} f(x) dx$?

(4-12) Which of the following statements is false?

(A) $\int_{-\pi}^{\pi} \sin x dx = 0$;
(B) $\int_0^1 \sqrt{1 - x^2} dx = - \int_1^0 \sqrt{1 - x^2} dx$;
(C) $\int_{-1}^{1} \frac{1}{x^2} dx = \left[-\frac{1}{x}\right]_{-1}^{1} = -1 - 1 = -2$.

(4-13) Which of the following statements is false?

(A) If f is continuous on $[0, 1]$, then $\int_0^1 f(x)dx = \int_0^1 f(1 - x)dx$;

(B) If $2 \le f(x) \le 4$ for $0 \le x \le 1$, then $1 < \int_0^1 f(x)dx < 5$;

(C) If f is continuous on $[a, b]$, then f is integrable on $[a, b]$.

(4-14) Suppose that f is continuous on $[1, 4]$ and $\int_1^4 f(x)dx = 10$. Which of the following statements is always true?

(A) The average of f on the interval $[1, 4]$ is equal to 3;

(B) The maximum value of f on the interval $[1, 4]$ is less than 4;

(C) The minimum value of f on the interval $[1, 4]$ is greater than 0;

(D) The maximum value of f on the interval $[1,4]$ is greater than 3.

(4-15) Evaluate the definite integral $\int_{-\pi/2}^{\pi/2} \frac{x^2 \tan x}{2+x^4} dx$.

(4-16) Evaluate the definite integral $\int_{-1}^{1} \frac{\tan x}{1+x^2} dx$.

(4-17) Find the derivative of $f(x) = \int_{x^2}^{4} 5\sqrt{1+t^2} dt$.

(4-18) Find the derivative of $f(x) = \int_{2}^{1/x} \sin^4 t dt$.

(4-19) Find the derivative of $\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(t)dt\right]$.

(4-20) Find $f(4)$ if $\int_0^{x^2} f(t)dt = x\cos(\pi x)$.

(4-21) Let $F(x) = \int_0^{x^3} \sin(t^2)dt$. Find $F'(x)$.

(4-22) The mass of part of a wire is $x(1+\sqrt{x})$ kilogram, where x is measured in meters from one end of the wire. Find the linear density of the wire when $x = 16$m.

(4-23) Suppose that f has a positive derivative for all values of x and that $f(1) = 0$. Let $g(x) = \int_0^x f(t)dt$. Which one of the following statements is false?

(A) g has a relative maximum at $x = 1$.

(B) g is a differentiable function.

(C) g has a horizontal tangent at $x = 1$.

(4-24) Suppose that $F(x) = \int_1^x f(t)dt$ with $f(t) = \int_1^{t^2} \frac{\sqrt{2+u^2}}{u} du$. Find $F''(2)$.

(4-25) On what interval is the curve $y = \int_0^x \frac{t^2}{t^2+t+2} dt$ concave downward?

解答

(4-1) **Ans.** $-\cos(x^7) + C$.
Let $y = x^7$. Then $dy = 7x^6 dx$. Hence $\int 7x^6 \sin x^7 dx = \int \sin y dy = -\cos y = -\cos(x^7) + C$.

(4-2) **Ans.** 6.
By definition, the tangent line of curve $y = f(x)$ at the point (x_0, y_0) is

$$y - y_0 = f'(x_0)(x - x_0) \Leftrightarrow y = f'(x_0)(x - x_0) + y_0 = f'(x_0)x + (y_0 - x_0 f'(x_0)).$$

Then by the condition that the slope of its tangent line at $(x, f(x))$ is $10x - 4$, we get that $f'(x_0) = 10x_0 - 4$ for all x_0. Equivalently, $f'(x) = 10x - 4$. Then $f(x) = 5x^2 - 4x + c$ for some constant c since $5x^2 - 4x + c$ is an antiderivative of

$10x-4$. Furthermore, by the condition that f passes through the point $(4,69)$, i.e., $f(4)=69$, we conclude that $f(4)=80-16+c=69$ or $c=5$. So $f(x)=5x^2-4x+5$ and hence $f(1)=6$.

(4-3) **Ans.** $\int_8^{10}\frac{6}{x^2}\,dx$.

Denote $\lim\limits_{|\triangle|\to 0}\sum\limits_{i=1}^{n}\frac{6}{c_i^2}\,\Delta x_i=\lim\limits_{|\triangle|\to 0}\sum\limits_{i=1}^{n}f(c_i)\,\Delta x_i$, where we define $f(x):=\frac{6}{x^2}$. So

$$\lim_{|\triangle|\to 0}\sum_{i=1}^{n}\frac{6}{c_i^2}\,\Delta x_i=\int_8^{10}f(x)\,dx=\int_8^{10}\frac{6}{x^2}\,dx.$$

(4-4) **Ans.** $\int_0^1 x^4\,dx$.

Write $\sum\limits_{i=1}^{n}\frac{i^4}{n^5}$ as

$$\sum_{i=1}^{n}\frac{i^4}{n^5}=\sum_{i=1}^{n}\frac{i^4}{n^4}\frac{1}{n}=\sum_{i=1}^{n}\left(\frac{i}{n}\right)^4\frac{1}{n}.$$

Hence

$$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i^4}{n^5}=\int_0^1 x^4\,dx.$$

(4-5) **Ans.** 4.

Since $2x-x^2=x(2-x)$ ans

$$2x-x^2\begin{cases}\geq 0 & \text{if } x\in[0,2],\\ <0 & \text{if } x\notin[0,2].\end{cases}$$

Hence

$$\begin{aligned}\int_{-1}^{3}|2x-x^2|dx&=\int_{-1}^{0}|2x-x^2|dx+\int_0^2|2x-x^2|dx+\int_2^3|2x-x^2|dx\\&=-\int_{-1}^{0}2x-x^2dx+\int_0^2 2x-x^2dx-\int_2^3 2x-x^2dx\\&=-\left[x^2-\frac{1}{3}x^3\right]_{x=-1}^{0}+\left[x^2-\frac{1}{3}x^3\right]_{x=0}^{2}-\left[x^2-\frac{1}{3}x^3\right]_{x=2}^{3}\\&=-\left\{[0]-\left[1+\frac{1}{3}\right]\right\}+\left\{\left[4-\frac{8}{3}\right]-[0]\right\}-\left\{[9-9]-\left[4-\frac{8}{3}\right]\right\}\\&=\left[1+\frac{1}{3}\right]+2\left[4-\frac{8}{3}\right]=4.\end{aligned}$$

(4-6) **Ans.** $\frac{1}{6}\left[21^{3/2}-6^{3/2}\right]$.

To compute $\int_1^2 x^3\sqrt{x^4+5}\,dx$, let $y=x^4+5$. Then $dy=(4x^3)dx$, and $y=6$ if $x=1$, and $y=21$ if $x=2$. Hence

$$\int_1^2 x^3\sqrt{x^4+5}\,dx=\int_6^{21}\frac{1}{4}\sqrt{y}\,dy=\frac{1}{4}\int_6^{21}y^{1/2}\,dy=\frac{1}{4}\left[\frac{2}{3}y^{\frac{3}{2}}\right]_{y=6}^{21}=\frac{1}{6}\left[21^{3/2}-6^{3/2}\right].$$

(4-7) **Ans.** $\frac{40}{3}$.

The average value of $f(x) = \frac{5(x^2+5)}{x^2}$ on the interval $[1,3]$ is equal to $\frac{\int_1^3 f(x)dx}{5-3}$. We compute

$$\begin{aligned}\int_1^3 f(x)dx &= \int_1^3 \frac{5(x^2+5)}{x^2}dx = 5\int_1^3 (1+5x^{-2})dx \\ &= 5\left[x - 5x^{-1}\right]_{x=1}^3 = 5\left[3 - \frac{5}{3} - 1 + 5\right] = \frac{80}{3}.\end{aligned}$$

Hence $\frac{\int_1^3 f(x)dx}{5-3} = \frac{40}{3}$.

(4-8) **Ans.** 4.

To compute $\int_0^\pi \sec^2(x/4)dx$, let $y = x/4$. Then $dy = dx/4$, and $y = 0$ as $x = 0$, and $y = \pi/4$ as $x = \pi$. Hence

$$\int_0^\pi \sec^2(x/4)dx = 4\int_0^{\pi/4} \sec^2(y)dy = 4\left[\tan(y)\right]_{y=0}^{\pi/4} = 4(1-0) = 4.$$

(4-9) **Ans.** 3.

By the definition of the integral and the graph of $f(x) = \sin(x)$, we have that $\int_0^{3\pi/2} |\sin(x)|dx = 3\int_0^{\pi/2} \sin(x)dx = 3[-\cos(x)]_{x=0}^{\pi/2} = 3[-0-(-1)] = 3$.

(4-10) **Ans.** $\frac{2}{3}$.

To compute $\int_{-\pi/2}^{\pi/2} \sin^2(x)\cos(x)dx$, let $y = \sin(x)$. Then $dy = \cos(x)dx$, and $y = -1$ as $x = -\pi/2$, and $y = 1$ as $x = \pi/2$. Hence

$$\int_{-\pi/2}^{\pi/2} \sin^2(x)\cos(x)dx = \int_{-1}^1 y^2dy = \frac{1}{3}[y^3]_{y=-1}^1 = \frac{2}{3}.$$

(4-11) **Ans.** Ture.

Let $y = x + h$. Then $dy = dx$ and $y = a + h$ as $x = a$, and $y = b + h$ as $x = b$. Hence

$$\int_a^b f(x+h)dx = \int_{a+h}^{b+h} f(y)dy = \int_{a+h}^{b+h} f(x)dx.$$

(4-12) **Ans.** (C) is false.

Note, (A) is true since $\sin x$ is odd. (B) is true by the definition. (C) is false since $\frac{1}{x^2}$ is not continuous on the whole interval $[-1,1]$.

(4-13) **Ans.** (B) is false.

Note, (A) is true by letting $y = 1 - x$ and by the method of change of variable directly. (B) is false, since $f(x)$ may not be integrable. For instance, let

$$f(x) = \begin{cases} 2 & x \in \mathbb{Q}, \\ 4 & x \in \mathbb{Q}^c. \end{cases}$$

Then such f is not integrable (Check yourself by computing the lower sum and the upper bound). Remark that the statement is true if f is assumed to be continuous on $[0,1]$. (C) is true.

(4-14) **Ans.** (D) is true.

Note, the answer for (A) is $\frac{10}{3}$. (B) may be false, e.g., letting

$$f(x) = \begin{cases} 10 & x \in [1,2], \\ 0 & x \in (2,4], \end{cases}$$

then $\int_1^4 f(x)dx = 10$ but the maximum value of f on the interval $[1,4]$ is 10. (C) may be false, e.g., letting

$$f(x) = \begin{cases} -1 & x \in [1,2], \\ 11 & x \in (2,3), \\ 0 & x \in (3,4], \end{cases}$$

then $\int_1^4 f(x)dx = 10$ but the minimum value of f on the interval $[1,4]$ is -1. (D) is true. If not, then $f(x) \le 3$. It follows that $10 = \int_1^4 f(x)dx \le \int_1^4 3dx = 9$, a contradiction.

(4-15) **Ans.** 0.

Since $f(x) = \frac{x^2 \tan x}{2+x^4} = \frac{x^2}{2+x^4} \tan x$ is a product of an even function and an odd function, f is also even. So $\int_{-\pi/2}^{\pi/2} \frac{x^2 \tan x}{2+x^4} dx = 0$.

(4-16) **Ans.** 0.

Since $f(x) = \frac{\tan x}{1+x^2} = \frac{1}{1+x^2} \tan x$ is a product of an even function and an odd function, f is also even. So $\int_{-1}^{1} \frac{\tan x}{1+x^2} dx = 0$.

(4-17) **Ans.** $-10x\sqrt{1+x^4}$.

We compute that

$$\begin{aligned}
\frac{d}{dx} f(x) &= \frac{d}{dx} \int_{x^2}^{4} 5\sqrt{1+t^2}dt \\
&= -\frac{d}{dx} \int_4^{x^2} 5\sqrt{1+t^2}dt \\
&= -\left[\frac{d}{du} \int_4^u 5\sqrt{1+t^2}dt\right]_{u=x^2} \left[\frac{d}{dx} x^2\right] \\
&= -5\left[\sqrt{1+u^2}\right]_{u=x^2} (2x) \\
&= -10x\sqrt{1+x^4}.
\end{aligned}$$

(4-18) **Ans.** $-x^{-2}\sin^4(x^{-1})$.

We compute that

$$\begin{aligned}
\frac{d}{dx} f(x) &= \frac{d}{dx} \int_2^{1/x} \sin^4 t dt \\
&= \frac{d}{dx} \int_2^{x^{-1}} \sin^4 t dt \\
&= \left[\frac{d}{du} \int_2^u \sin^4 t dt\right]_{u=x^{-1}} \left[\frac{d}{dx} x^{-1}\right] \\
&= \sin^4(x^{-1})(-x^{-2}) = -x^{-2}\sin^4(x^{-1}).
\end{aligned}$$

(4-19) **Ans.** $f(v(x))v'(x) - f(u(x))v'(x)$.
We compute that

$$\frac{d}{dx}\left[\int_{u(x)}^{v(x)} f(t)dt\right] = \frac{d}{dx}\left[\int_0^{v(x)} f(t)dt - \int_0^{u(x)} f(t)dt\right]$$
$$= f(v(x))v'(x) - f(u(x))v'(x).$$

(4-20) **Ans.** $1/4$.
We compute that

$$\int_0^{x^2} f(t)dt = x\cos(\pi x) \Rightarrow f(x^2)(2x) = \cos(\pi x) - x\pi\sin(\pi x).$$

Putting $x = 2$ into the above equality, we have that $4f(4) = \cos(2\pi) - 2\pi\sin(2\pi) = 1$. So $f(4) = \frac{1}{4}$.

(4-21) **Ans.** $3x^2\sin(x^6)$.
Let $F(x) = \int_0^{x^3}\sin(t^2)dt$. We compute that $F'(x) = \sin((x^3)^2)(3x^2) = 3x^2\sin(x^6)$.

(4-22) **Ans.** 7kg/m.
Let the linear density of the wire at point x be $f(x)$. Then the assumption that the mass of part of a wire is $x(1+\sqrt{x})$ kilogram implies that

$$\int_0^x f(t)dt = x(1+\sqrt{x}).$$

It then implies that

$$f(x) = \frac{d}{dx}[x(1+\sqrt{x})] \quad = \frac{d}{dx}(x + x^{3/2})$$
$$= 1 + \frac{3}{2}x^{1/2}.$$

Hence $f(16) = 1 + \frac{3}{2}4 = 7$.

(4-23) **Ans.** (A) is false.
By the condition f has a positive derivative for all values of x, we have that f is continuous and $f'(x) > 0$. So f is integrable. Let $g(x) = \int_0^x f(t)dt$. Then we have that $g'(x) = f(x)$, so g is differentiable, i.e., (B) is true. Moreover, since $g'(1) = f(1) = 0$, we have that g has a horizontal tangent at $x = 1$, i.e., (C) is true. Finally, we compute that $g''(x) = f(x) > 0$. Then $g'(1) = 0$ and $g''(x) > 0$ imply that g has a relative minimum at $x = 1$. So (A) is false.

(4-24) **Ans.** $3\sqrt{2}$.
Let $F(x) = \int_1^x f(t)dt$. Then we have that $F'(x) = f(x)$. So $F''(x) = f'(x)$. Moreover, since $f(t) = \int_1^{t^2}\frac{\sqrt{2+u^2}}{u}du$, we have that $f'(t) = 2t\frac{\sqrt{2+t^4}}{t^2}$. Hence $F''(x) = 2x\frac{\sqrt{2+x^4}}{x^2}$ and $F''(2) = 4\frac{\sqrt{18}}{4} = \sqrt{18} = 3\sqrt{2}$.

(4-25) **Ans.** $(-4, 0)$.

Let $y = \int_0^x \frac{t^2}{t^2+t+2} dt$. We compute that $y' = \frac{x^2}{x^2+x+2}$ and

$$y'' = \frac{2x(x^2+x+2) - x^2(2x+1)}{(x^2+x+2)^2} = \frac{x^2+4x}{(x^2+x+2)^2} = \frac{x(x+4)}{(x^2+x+2)^2}$$

So $y'' < 0$ on $(-4, 0)$. That is, curve $y = \int_0^x \frac{t^2}{t^2+t+2} dt$ is concave downward on $(-4, 0)$.

第五章（超越函數）題目

(5-1) Evaluate $\lim\limits_{n\to\infty} \frac{1}{n}\sum\limits_{i=1}^{n} \frac{1}{1+(i/n)}$.

(5-2) Find the equation of the tangent line to the curve $y = x^{\cos x}$ at $x = \pi/2$.

(5-3) Let $F(x) = \int_{x^3}^{1} \sqrt{14+2^t}\,dt$. Find the derivative $(F^{-1})'(0)$.

(5-4) Find $\int_1^e \frac{\ln(x^2)}{x}dx$.

(5-5) Let $f(x) = \int_1^x \frac{\exp(-\sqrt{t})}{\sqrt{t}}dt$. Evaluate $\lim_{x\to\infty} f(x) =$?

(5-6) Find $\frac{dy}{dx}$ where $xe^y - xy + 3y^2 = 0$.

(5-7) Solve x for the equation $\arccos(\sqrt{3x}) = \arcsin(\sqrt{x})$.

(5-8) For what values of a does the curve $y = ax^3 + e^x$ have no inflection point?
(A) -1, (B) 0, (C) 1, (D) 2.

(5-9) Let $f(x) = \frac{1}{8}x - 3$ and $g(x) = x^3$. Find $(f^{-1} \circ g^{-1})(1)$.

(5-10) If $f(x) = x^n$, then does f^{-1} exist?

(5-11) Let $f(x) = \dfrac{e^x - e^{-x}}{2}$. Then $f^{-1}(x) =$?

解答

(5-1) **Ans.** $\int_0^1 \frac{1}{1+x}dx = \ln(2)$.
Write

$$\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+(i/n)} = \sum_{i=1}^{n} f(c_i)\Delta x_i$$

where we define $c_i = \frac{i}{n}$, $f(x) = \frac{1}{1+x}$, $\Delta x_i = \frac{1}{n}$. Hence

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n}\frac{1}{1+(i/n)} = \int_0^1 \frac{1}{1+x}dx = [\ln|1+x|]_{x=0}^{1} = \ln(2) - \ln(1) = \ln(2).$$

(5-2) **Ans.** $y - 1 = -(\ln\frac{\pi}{2})(x - \frac{\pi}{2})$.
From the equation $y = x^{\cos x} = e^{(\ln x)(\cos x)}$, we compute that

$$y' = e^{(\ln x)(\cos x)} \cdot \left(\frac{1}{x}\cos x - (\ln x)(\sin x)\right)$$

$$\Leftrightarrow y' = x^{\cos x} \cdot \left(\frac{1}{x}\cos x - (\ln x)(\sin x)\right)$$

Hence the slope of the graph of $y = x^{\cos x}$ at $x = \pi/2$ is

$$\left(\frac{\pi}{2}\right)^{\cos\frac{\pi}{2}} \cdot \left(\frac{1}{\frac{\pi}{2}}\cos\frac{\pi}{2} - \left(\ln\frac{\pi}{2}\right)\left(\sin\frac{\pi}{2}\right)\right) = (\frac{\pi}{2})^0 \cdot \left(\frac{2}{\pi}\cdot 0 - \left(\ln\frac{\pi}{2}\right)\cdot 1\right) = -\ln\frac{\pi}{2}.$$

It follows that the equation of the tangent line to the curve $y = x^{\cos x}$ at $x = \pi/2$ is $y - 1 = -(\ln \frac{\pi}{2})(x - \frac{\pi}{2})$ since $1^{\cos 1} = 1$.

(5-3) **Ans.** $-\frac{1}{12}$.
Let $F(x) = \int_{x^3}^{1} \sqrt{14 + 2^t}dt$. We compute that $F'(x) = -3x^2\sqrt{14 + 2^{(x^3)}}$. On the other hand, since $F(F^{-1}(x)) = x$, we have that $F'(F^{-1}(x))(F^{-1})'(x) = 1$. That is,

$$(F^{-1})'(x) = \frac{1}{F'(F^{-1}(x))}.$$

So

$$(F^{-1})'(0) = \frac{1}{F'(F^{-1}(0))}.$$

Moreover, since $F(1) = \int_{1}^{1} \sqrt{14 + 2^t}dt = 0$, we have that $F^{-1}(0) = 1$. Then, from above equality,

$$(F^{-1})'(0) = \frac{1}{F'(1)} = \frac{1}{-3\sqrt{14+2}} = -\frac{1}{12}.$$

(5-4) **Ans.** 1.
Let $y = \ln x$. Then $dy = \frac{1}{x}dx$, and $y = 0$ if $x = 1$, and $y = 1$ if $x = e$. Hence

$$\int_1^e \frac{\ln(x^2)}{x}dx = 2\int_1^e \frac{\ln(x)}{x}dx = 2\int_0^1 ydy = [y^2]_{y=0}^1 = 1.$$

(5-5) **Ans.** $2\exp(-1)$.
Let $y = \sqrt{t}$. Then $dy = \frac{1}{2\sqrt{t}}dt$, and $y = 1$ if $t = 1$, and $y = \sqrt{x}$ if $t = x$. Hence

$$\int_1^x \frac{\exp(-\sqrt{t})}{\sqrt{t}}dt = 2\int_1^{\sqrt{x}} \exp(-y)dy = 2[-\exp(-y)]_{y=1}^{\sqrt{x}} = 2[-\exp(-\sqrt{x})+\exp(-1)]$$

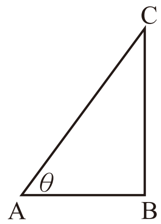
Hence

$$\lim_{x\to\infty} \int_1^x \frac{\exp(-\sqrt{t})}{\sqrt{t}}dt = \lim_{x\to\infty} 2[-\exp(-\sqrt{x}) + \exp(-1)] = 2\exp(-1).$$

(5-6) **Ans.** $y' = -\frac{e^y - y}{xe^y - x + 6y}$.
By direct computation,

$$\begin{aligned}
&\frac{d}{dx}[xe^y - xy + 3y^2] = \frac{d}{dx}0. \\
&\Rightarrow e^y + xe^y y' - y - xy' + 3(2y)y' = 0. \\
&\Rightarrow (xe^y - x + 6y)y' + (e^y - y) = 0 \\
&\Rightarrow y' = -\frac{e^y - y}{xe^y - x + 6y}.
\end{aligned}$$

(5-7) **Ans.** $x = 1/4$.
Let $\overline{AC} = L$. Then $\arccos(\sqrt{3x})$ implies that $\overline{AB} = \sqrt{3x}L$, while $\arcsin(\sqrt{x})$ implies that $\overline{BC} = \sqrt{x}L$. Since $\overline{AB}^2 + \overline{BC}^2 = \overline{AC}^2$, we have that $3xL^2 + xL^2 = L^2$. So $4xL^2 = L^2$, i.e., $x = 1/4$.



(5-8) **Ans.** (B).

We compute that $y' = 3ax^2 + e^x$ and $y'' = 6ax + e^x$. Note that, as $a = 0$, $6ax + e^x = e^x > 0$ for all x, which implies that $y = ax^3 + e^x$ have no inflection point. Hence (B) is true.

(5-9) **Ans.** 32.

$$\begin{aligned}(f^{-1} \circ g^{-1})(1) &= f^{-1}(g^{-1}(1)) \\ &= f^{-1}(1) \quad (\because g(1) = 1. \quad \therefore g^{-1}(1) = 1.) \\ &= 32 \quad (\because f(32) = 1. \quad \therefore f^{-1}(1) = 32.)\end{aligned}$$

(5-10) **Ans.** No.

Because when n is an even number, $f(x) = x^n = (-x)^n = f(-x)$ for all $x \in \mathbb{R}$. It implies that f is not one-to-one and hence f^{-1} doesn't exist.

(5-11) **Ans.** $f^{-1}(x) = \ln(x + \sqrt{x^2 + 1})$.

Let $y = f(x) = \frac{e^x - e^{-x}}{2}$. Then

$$f^{-1}(y) = x. \tag{6}$$

Moreover,

$$\begin{aligned} & y = \frac{e^x - e^{-x}}{2} \\ & \Rightarrow e^x - e^{-x} = 2y \quad \Rightarrow e^x - \frac{1}{e^x} = 2y \\ & \Rightarrow e^{2x} - 1 = 2ye^x \quad (\text{multiply } e^x \text{ on both sides of } =.) \\ & \Rightarrow (e^x)^2 - 2y\,(e^x) - 1 = 0 \\ & \Rightarrow e^x = \frac{2y \pm \sqrt{(2y)^2 + 4}}{2} = y \pm \sqrt{y^2 + 1} \\ & \Rightarrow e^x = y + \sqrt{y^2 + 1} \quad (\because y - \sqrt{y^2 + 1} < 0.) \\ & \Rightarrow x = \ln(y + \sqrt{y^2 + 1}) \end{aligned} \tag{7}$$

By (6) and (7), we have that $f^{-1}(y)\,(= x) = \ln(y + \sqrt{y^2 + 1})$.