

國立高雄大學應用數學系
每月挑戰 019 解答
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2013年4月

答題優良名單： 以下是連威翔同學的解答。

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$$(1) \frac{e^{ixt}}{1+t^2} = \frac{e^{ixt}}{(t+i)(t-i)}. \text{ By residue theorem,}$$

$$\text{Res}_{t=i} \frac{e^{ixt}}{1+t^2} = \lim_{t \rightarrow i} \frac{e^{ixt}(t-i)}{(t+i)(t-i)} = \lim_{t \rightarrow i} \frac{e^{ixt}}{t+i} = \frac{e^{-x}}{2i}, \text{ where 'Res' means residue, and}$$

$$J(x) = \oint_C \frac{e^{ixt}}{1+t^2} dt = 2\pi i \times \frac{e^{-x}}{2i} = \pi e^{-x}.$$

(2) Choose a real number $R > 1$ and let $K_R = \{R e^{i\theta} \mid 0 \leq \theta \leq \pi\}$. Choose the contour $C = C_R = \{-R < t < R\} \cup K_R$, then C enclose i .

$$\text{Integrate } \frac{e^{ixt}}{1+t^2} \text{ along } C, \text{ then } J(x) = \oint_C \frac{e^{ixt}}{1+t^2} dt = \int_{-R}^R \frac{e^{ixt}}{1+t^2} dt + \int_{K_R} \frac{e^{ixt}}{1+t^2} dt.$$

The first integral:

$$\int_{-R}^R \frac{e^{ixt}}{1+t^2} dt = \int_{-R}^R \frac{\cos xt + i \sin xt}{1+t^2} dt = \int_{-R}^R \frac{\cos xt}{1+t^2} dt + i \int_{-R}^R \frac{\sin xt}{1+t^2} dt = 2 \int_0^R \frac{\cos xt}{1+t^2} dt$$

because $\frac{\cos xt}{1+t^2}$ is an even function and $\frac{\sin xt}{1+t^2}$ is odd.

Let R tends to infinity, then $\int_{-R}^R \frac{e^{ixt}}{1+t^2} dt = 2 \int_0^\infty \frac{\cos xt}{1+t^2} dt$ and

$$\left| \int_{K_R} \frac{e^{ixt}}{1+t^2} dt \right| \leq \int_{K_R} \left| \frac{e^{ixt}}{1+t^2} \right| |dt| = \int_{K_R} \frac{1}{|1+t^2|} |dt| \leq \frac{\pi R}{R^2-1} \rightarrow 0$$

Handwritten note: $t = R e^{i\theta}$ has a nonzero imaginary part, so $|e^{ixt}| \neq 1$. See solution later.

because K_R has arc-length πR , $|e^{ixt}| = 1$,

$$\text{and } |1+t^2| \geq |t^2| - 1 = |t|^2 - 1 = R^2 - 1 > 0 \Rightarrow \frac{1}{|1+t^2|} \leq \frac{1}{R^2-1}, \text{ for any } t \in K_R.$$

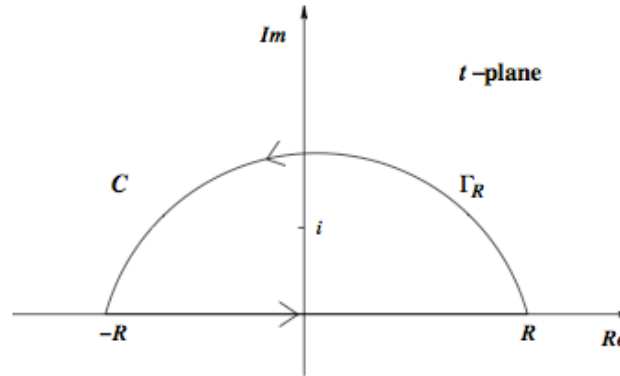
$$\text{So } J(x) = 2 \int_0^\infty \frac{\cos xt}{1+t^2} dt \Rightarrow \int_0^\infty \frac{\cos xt}{1+t^2} dt = \frac{J(x)}{2}.$$

以下是我的解答。

Consider contour integral

$$J(x) = \oint_C \frac{e^{\iota x t}}{1+t^2} dt,$$

with the contour C below



We will evaluate the integral (i) by Residue theorem and (ii) in sections.

(i) Residue theorem gives, with the contour C enclosing the singularity $t = i$ only,

$$J(x) = 2\pi\iota \times \text{Residue}_{t=\iota} \left(\frac{e^{\iota x t}}{1+t^2} \right),$$

and since $t = \iota$ is a simple pole,

$$J(x) = 2\pi\iota \times \lim_{t \rightarrow \iota} \frac{e^{\iota x t}}{1+t^2} (t - \iota) = \frac{2\pi\iota e^{-x}}{2\iota} = \pi e^{-x}.$$

(ii) Evaluate $J(x)$ in sections

$$J(x) = \int_{-R}^R \frac{e^{\iota x t}}{1+t^2} dt + \int_{\Gamma_R} \frac{e^{\iota x t}}{1+t^2} dt.$$

Taking the limit $R \rightarrow \infty$,

$$\begin{aligned} \lim_{R \rightarrow \infty} \int_{-R}^R \frac{e^{\iota x t}}{1+t^2} dt &= \int_{-\infty}^{\infty} \frac{e^{\iota x t}}{1+t^2} dt \\ &= \int_{-\infty}^0 \frac{e^{\iota x t}}{1+t^2} dt + \int_0^{\infty} \frac{e^{\iota x t}}{1+t^2} dt \\ \text{put } t = -s, &= \int_0^{\infty} \frac{e^{-\iota x s}}{1+s^2} ds + \int_0^{\infty} \frac{e^{\iota x t}}{1+t^2} dt \\ &= \int_0^{\infty} \frac{e^{-\iota x t} + e^{\iota x t}}{1+t^2} dt = 2 \int_0^{\infty} \frac{\cos xt}{1+t^2} dt. \end{aligned}$$

On Γ_R , $t = Re^{\iota\theta}$ with $0 < \theta < \pi$. Thus $e^{\iota x t} = e^{\iota x R e^{\iota\theta}} = e^{\iota x R (\cos \theta + \iota \sin \theta)}$,

$$\lim_{R \rightarrow \infty} \int_{\Gamma_R} \frac{e^{\iota x t}}{1+t^2} dt = \lim_{R \rightarrow \infty} \int_0^{\pi} \frac{e^{\iota x R (\cos \theta + \iota \sin \theta)}}{1+R^2 e^{2\iota\theta}} R \iota e^{\iota\theta} d\theta,$$

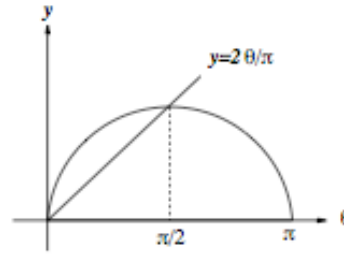
and

$$\left| \int_{\Gamma_R} \right| \leq \int_0^\pi \left| \frac{e^{-xR \sin \theta}}{1 + R^2 e^{2i\theta}} \right| R d\theta.$$

Since R is sufficiently large, we have

$$\left| 1 + R^2 e^{2i\theta} \right| \geq \left| |1| - |R^2 e^{2i\theta}| \right| = R^2 - 1.$$

Thus,



$$\begin{aligned} \left| \int_{\Gamma_R} \right| &\leq \frac{2R}{R^2 - 1} \int_0^{\frac{\pi}{2}} e^{-xR \sin \theta} d\theta, \\ &\leq \frac{2R}{R^2 - 1} \int_0^{\frac{\pi}{2}} e^{-xR \frac{2\theta}{\pi}} d\theta, \quad \sin \theta \geq \frac{2\theta}{\pi}, \quad 0 \leq \theta \leq \frac{\pi}{2}, \\ &= \frac{2R}{R^2 - 1} \left[\frac{e^{-xR \frac{2\theta}{\pi}}}{-\frac{2}{\pi} x R} \right]_0^{\frac{\pi}{2}} \\ &= \frac{2R}{R^2 - 1} \cdot \frac{\pi}{2xR} (1 - e^{-Rx}). \end{aligned}$$

$$\left| \int_{\Gamma_R} \right| \leq \frac{\pi}{R^2 - 1} \cdot \frac{1 - e^{-Rx}}{x} \rightarrow 0 \quad \text{as } R \rightarrow \infty.$$

Hence taking limit as $R \rightarrow \infty$,

$$J(x) = 2 \int_0^\infty \frac{\cos xt}{1 + t^2} dt.$$

Combining results

$$2 \int_0^\infty \frac{\cos xt}{1 + t^2} dt = \pi e^{-x} \quad \Rightarrow \quad \int_0^\infty \frac{\cos xt}{1 + t^2} dt = \frac{\pi}{2} e^{-x}.$$