國立高雄大學應用數學系

每月挑戰 024 解答

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2013 年10 月

1. Let $A \in M_{n\times n}(\mathbb{R})$ be nonsingular and $v \in \mathbb{R}^n$. Show that

$$\det(A + vv^T) = \det(A)\left(1 + v^T A^{-1} v\right).$$

Ans. Since

$$\begin{bmatrix} I_n & 0 \\ v^T & 1 \end{bmatrix}\begin{bmatrix} I_n + A^{-1}vv^T & A^{-1}v \\ 0 & 1 \end{bmatrix}\begin{bmatrix} I_n & 0 \\ -v^T & 1 \end{bmatrix} = \begin{bmatrix} I_n & A^{-1}v \\ 0 & 1 + v^T A^{-1} v \end{bmatrix},$$

we have $\det\left(I_n + A^{-1}vv^T\right) = 1 + v^T A^{-1} v$.
Thus, $\det(A^{-1})\ \det(A + vv^T) = 1 + v^T A^{-1} v$.
That is,

$$\det(A + vv^T) = \det(A)(1 + v^T A^{-1} v).$$

答題優良名單：博士班連威翔同學。

以下是連威翔同學的解答。

Let $A \in M_{n\times n}(R)$ be nonsingular and $v \in R^n$. Show that

$$\det(A + vv^T) = \det(A)(1 + v^T A^{-1} v).$$

[Solution]:

We express A as $\begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix}$, where $w_1, w_2, \ldots, w_n$ are row vectors. Let $v = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ and

use the same way to express $vv^T = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}\begin{bmatrix} b_1 & b_2 & \cdots & b_n \end{bmatrix} = \begin{bmatrix} b_1 v^T \\ b_2 v^T \\ \vdots \\ b_n v^T \end{bmatrix}$.

If $v = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$ (zero vector), then $vv^T = O_{n\times n}$, and $v^T A^{-1} v = v^T \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = 0$. So we have

$\det(A + vv^T) = \det(A) = \det(A)(1 + v^T A^{-1} v)$.

Otherwise, v has at least one nonzero component, say $b_j \neq 0$. So

$$\det(A+vv^T)=\begin{vmatrix} w_1+b_1v^T \\ \vdots \\ w_j+b_jv^T \\ \vdots \\ w_n+b_nv^T \end{vmatrix}=\begin{vmatrix} w_1-\frac{b_1}{b_j}w_j \\ \vdots \\ w_j+b_jv^T \\ \vdots \\ w_n-\frac{b_n}{b_j}w_j \end{vmatrix}$$, where we did $(n-1)$ row operations:

$R_i - R_j \times (\frac{b_i}{b_j}), 1 \le i \le n, i \ne j$ at the second equality. Using the property of determinants, we have

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$$\det(A+vv^T)=\begin{vmatrix} w_1-\frac{b_1}{b_j}w_j \\ \vdots \\ w_j \\ \vdots \\ w_n-\frac{b_n}{b_j}w_j \end{vmatrix}+\begin{vmatrix} w_1-\frac{b_1}{b_j}w_j \\ \vdots \\ b_jv^T \\ \vdots \\ w_n-\frac{b_n}{b_j}w_j \end{vmatrix}=\begin{vmatrix} w_1 \\ \vdots \\ w_j \\ \vdots \\ w_n \end{vmatrix}+\begin{vmatrix} w_1-\frac{b_1}{b_j}w_j \\ \vdots \\ b_jv^T \\ \vdots \\ w_n-\frac{b_n}{b_j}w_j \end{vmatrix}=\det(A)+\begin{vmatrix} w_1-\frac{b_1}{b_j}w_j \\ \vdots \\ b_jv^T \\ \vdots \\ w_n-\frac{b_n}{b_j}w_j \end{vmatrix}\cdots(1)$$

Since A is nonsingular, the row vectors $w_1, w_2, \ldots, w_n$ of A form a basis of R^n.

So we can find $\alpha_1, \alpha_2, \ldots, \alpha_n \in R$ such that $v^T = \alpha_1 w_1 + \alpha_2 w_2 + \ldots + \alpha_n w_n$. For the last determinat of (1), now we have

$$\begin{vmatrix} w_1-\frac{b_1}{b_j}w_j \\ \vdots \\ b_jv^T \\ \vdots \\ w_n-\frac{b_n}{b_j}w_j \end{vmatrix}=(\frac{1}{b_j})^{n-1}\begin{vmatrix} b_jw_1-b_1w_j \\ \vdots \\ b_jv^T \\ \vdots \\ b_jw_n-b_nw_j \end{vmatrix}=(\frac{1}{b_j})^{n-1}\begin{vmatrix} b_jw_1-b_1w_j \\ \vdots \\ b_j\alpha_1w_1+b_j\alpha_2w_2+\ldots+b_j\alpha_jw_j+\ldots+b_j\alpha_nw_n \\ \vdots \\ b_jw_n-b_nw_j \end{vmatrix}$$

$$=(\frac{1}{b_j})^{n-1}\begin{vmatrix} b_jw_1-b_1w_j \\ \vdots \\ \alpha_jb_jw_j+\alpha_1b_1w_j+\ldots+\alpha_{j-1}b_{j-1}w_j+\alpha_{j+1}b_{j+1}w_j+\ldots+\alpha_nb_nw_j \\ \vdots \\ b_jw_n-b_nw_j \end{vmatrix}$$

$$=(\frac{1}{b_j})^{n-1}\begin{vmatrix} b_jw_1-b_1w_j \\ \vdots \\ (\alpha_1b_1+\ldots+\alpha_jb_j+\ldots+\alpha_nb_n)w_j \\ \vdots \\ b_jw_n-b_nw_j \end{vmatrix}=(\alpha_1b_1+\ldots+\alpha_jb_j+\ldots+\alpha_nb_n)(\frac{1}{b_j})^{n-1}\begin{vmatrix} b_jw_1-b_1w_j \\ \vdots \\ w_j \\ \vdots \\ b_jw_n-b_nw_j \end{vmatrix}\cdots(2)$$

where we did $(n-1)$ row operations: $R_j - \alpha_i R_i, 1 \le i \le n, i \ne j$ at the third equality.

To continue (2), we have

$$(\frac{1}{b_j})^{n-1}\begin{vmatrix} b_j w_1 - b_1 w_j \\ \vdots \\ w_j \\ \vdots \\ b_j w_n - b_n w_j \end{vmatrix} = (\frac{1}{b_j})^{n-1}\begin{vmatrix} b_j w_1 \\ \vdots \\ w_j \\ \vdots \\ b_j w_n \end{vmatrix} = \begin{vmatrix} w_1 \\ \vdots \\ w_j \\ \vdots \\ w_n \end{vmatrix} = \det(A)\cdots(3).$$

On the other hand, assume that

$A^{-1} = [u_1 \quad u_2 \quad \cdots \quad u_n]$, where $u_1, u_2, \ldots, u_n$ are column vectors. By $AA^{-1} = I$, we have $w_i u_i = \delta_{ij}$, where $1 \le i, j \le n$ and $\delta_{ij} = 1$ if $i = j$, $\delta_{ij} = 0$ if $i \ne j$. Now,

$v^T u_i = (\alpha_1 w_1 + \alpha_2 w_2 + \ldots + \alpha_n w_n)u_i = \alpha_i w_i u_i = \alpha_i$, where $1 \le i \le n$, and

$$\alpha_1 b_1 + \ldots + \alpha_j b_j + \ldots + \alpha_n b_n = [\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_n]\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = [v^T u_1 \quad v^T u_2 \quad \cdots \quad v^T u_n]\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

$$= v^T [u_1 \quad u_2 \quad \cdots \quad u_n]\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix} = v^T A^{-1} v \cdots (4).$$

By (2),(3),(4), we have $\begin{vmatrix} w_1 - \frac{b_1}{b_j} w_j \\ \vdots \\ b_j v^T \\ \vdots \\ w_n - \frac{b_n}{b_j} w_j \end{vmatrix} = \det(A) v^T A^{-1} v$. Put this back to (1), we have

$\det(A + vv^T) = \det(A) + \det(A) v^T A^{-1} v = \det(A)(1 + v^T A^{-1} v)$ and complete the proof.

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